#### The Domatic Numbers of Computable Graphs

Tyler Markkanen

Manhattan College

<span id="page-0-0"></span>CUNY Logic Workshop – May 9, 2014

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#### Joint with Matthew Jura and Oscar Levin.

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### Domatic Partitions

Domatic 2-partition

No domatic 3-partition



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### Domatic Partitions





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### Domatic Partitions





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### Domatic Partitions





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### Domatic Partitions





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### Domatic Partitions





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### Domatic Partitions





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## Domatic Partitions





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### Domatic Partitions





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### Domatic Partitions





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## Domatic Partitions



Domatic number of a graph  $G$ :

 $d(G)$  = the max n s.t. G has a domatic n-partition

Computable domatic number of a graph  $G$ :  $d^c(G) =$  $d^c(G) =$  $d^c(G) =$  the m[a](#page-22-0)x  $n$  s.[t](#page-32-0).  $\overline{G}$  $\overline{G}$  $\overline{G}$  has a computabl[e d](#page-13-0)[om](#page-15-0)a[ti](#page-2-0)c  $n$ [-p](#page-21-0)artit[io](#page-33-0)n



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## Separating the Domatic the Numbers

Let  $\varphi_0, \varphi_1, \varphi_2, \ldots$  list all partial computable functions  $\mathbb{N} \to \mathbb{N}$ . Example

There is a computable graph  $G$  such that  $d(G) = 3$  but  $d^c(G) < 3$ .

 $\rightarrow$ 

The gadget of  $\varphi_e$ :

Springing the trap:





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 $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ 

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# $d(G) - d^c(G)$  for Highly Computable Graphs

#### Definition

A graph  $G = (V, E)$  is **highly computable** if V and E are computable sets and there is a computable function that, when given  $v \in V$ , outputs the degree of v (i.e., the number of vertices adjacent to  $v$ ).

#### Theorem

For every  $n \geq 3$ , there is a highly computable graph G such that  $d(G) = n$  and  $d^c(G) = n - 1$ .



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 $\mathcal{A} \subseteq \mathcal{A} \Rightarrow \mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \subseteq \mathcal{B}$ 

 $\equiv$ 

 $\mathcal{O} \subset \mathcal{O}$ 

The  $K_n^-$ -Gadget





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 $d^c(G) < n$ 

Figure : Trapping a purported computable domatic  $n$ -partition.



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 $d(G) = n$ 



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 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$ 

 $\mathcal{O} \subset \mathcal{O}$ 

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## An Alternative Trap

Figure : Completing a  $K_n^-$  Loop.



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<span id="page-28-0"></span> $\mathcal{O} \subset \mathcal{O}$ 



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<span id="page-29-0"></span> $\mathcal{O} \subset \mathcal{O}$ 



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## A Graph's Minimal Degree

#### Definition

The **minimal degree** of a graph G, denoted by  $\delta(G)$ , is the minimal degree of the vertices of  $G$ .

#### Theorem (A version of a graph by Zelinka [\[2\]](#page-46-0))

There exists a computable graph G such that  $\delta(G) = 4$ ,  $d(G) = 3$ , and  $d^c(G) = 2$ .



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## A Zelinka-Type Construction

Figure : Example for  $\delta(G) = 4$  and  $d(G) = 3$ .



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## The Union of Zelinka's Graph with Another

#### Theorem

There exists a computable graph G such that  $\delta(G) = 4$ ,  $d(G) = 3$ , and  $d^c(G) = 2$ .

#### Proof.

Let  $G = G_1 \cup G_2$ , where

- $\delta(G_1) = 4$  and  $d(G_1) = 3$  (i.e.,  $G_1$  is our Zelinka-type graph);
- $G_2$  is computable such that  $d(G_2) = 5$  (so  $\delta(G_2) \geq 4$ ) and  $d^c(G_2) = 2.$



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## Total Domatic Partitions

#### Definition

For any  $n \geq 1$  and any graph  $G = (V, E)$ , a (computable) partition  $p: V \to \{1, \ldots, n\}$  into n colors is a (computable) total **domatic** *n*-partition if the vertices adjacent to v use up all *n* colors (i.e.,  $(\forall v \in V)(\forall i \in \{1, ..., n\})(\exists u \in V)[uEv \wedge p(u) = i]).$ 

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## The Double  $K_4$ -Gadget





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## The Regular 3-2 Problem

#### Definition

For any  $n$ , a graph is  $n$ -regular if the degree of every vertex is  $n$ .

#### Theorem

Every computable 3-regular graph that has a total domatic 3-partition has a computable total domatic 2-partition.

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## Double Coloring

Vertices on the border of two sets  $V_0$  and  $V_1$  will be doubly colored. Each doubly colored vertex  $(V_0)$ 's Color Choice $/V_1$ 's Color Choice) will be resolved by the rules: If red is present (as either set's choice), choose red. Otherwise, choose blue.





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Figure : Three Cases.





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## Future Research

#### **Conjecture**

Every highly computable graph with a domatic 4-partition has a computable domatic 3-partition.

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## Questions?

Thank you.

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M. Jura, O. Levin, T. Markkanen, Domatic partitions of computable graphs. Archive for Mathematical Logic, Volume 53, Issue 1 (2014), 137–155, DOI 10.1007/s00153-013-0359-2.

<span id="page-46-0"></span>譶 B. Zelinka, Domatic number and degrees of vertices of a graph. Mathematica Slovaca, Vol. 33 (1983), No. 2, 145–147.

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