The Domatic Numbers of Computable Graphs

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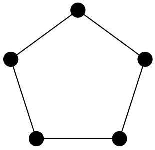
Manhattan College

CUNY Logic Workshop – May 9, 2014

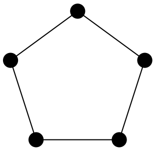
Joint with Matthew Jura and Oscar Levin.

Domatic Partitions

Domatic 2-partition



No domatic 3-partition



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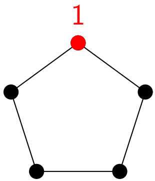

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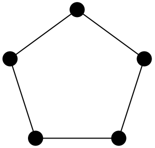

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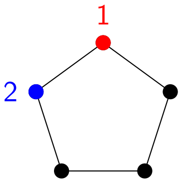

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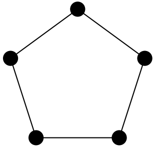

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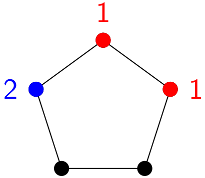

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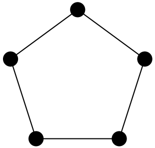

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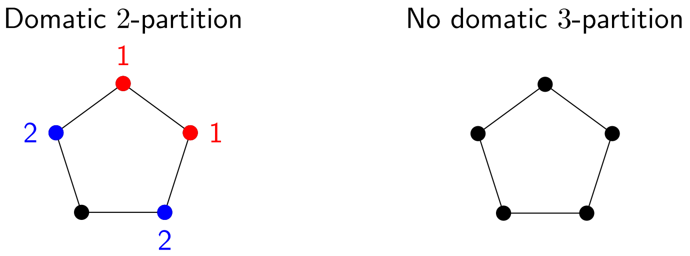

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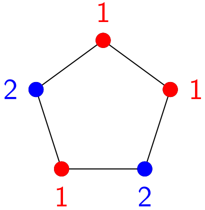

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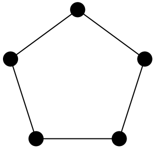

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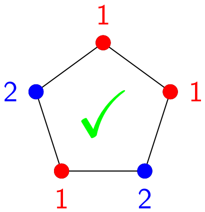

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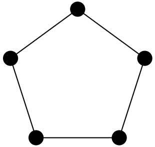

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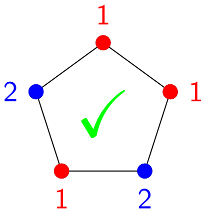

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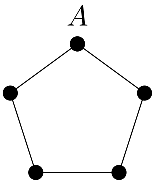

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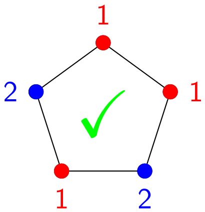

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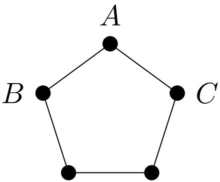

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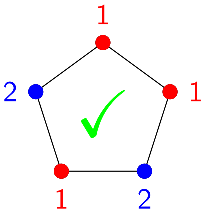

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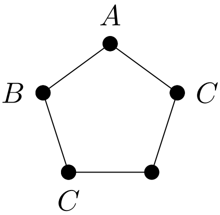

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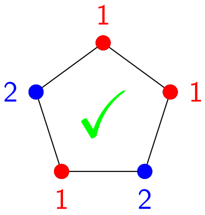

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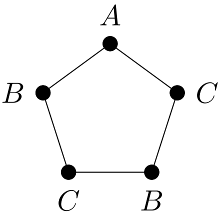

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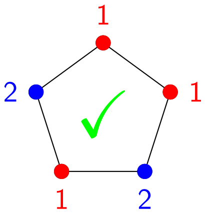

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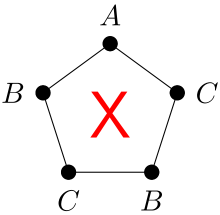

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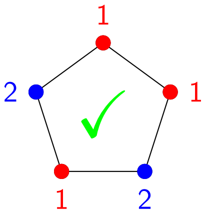

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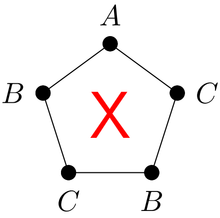

$$d(G) = 2$$

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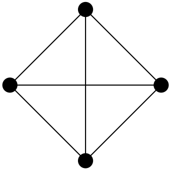
Separating the Domatic the Numbers

Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ list all partial computable functions $\mathbb{N} \to \mathbb{N}$.

Example

There is a computable graph G such that $d(G) = 3$ but $d^c(G) < 3$.

The gadget of φ_e:



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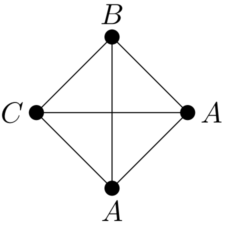

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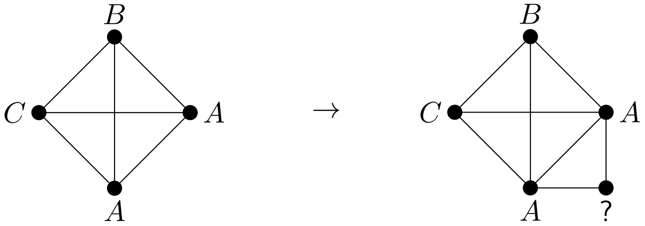

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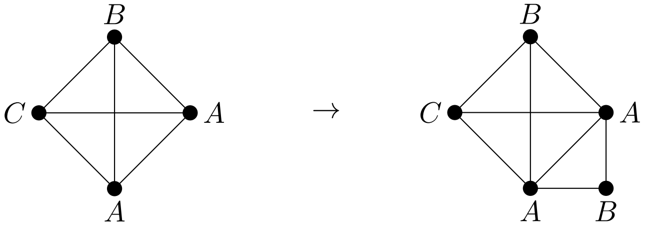

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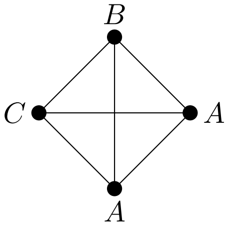

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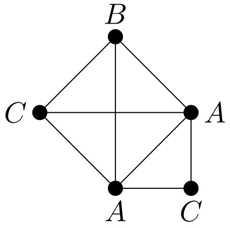

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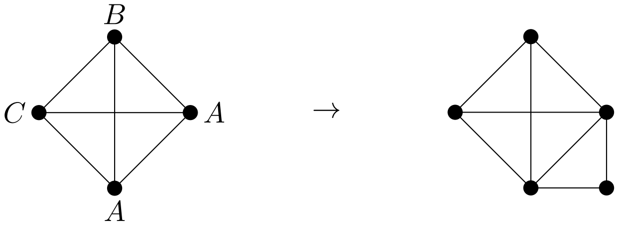

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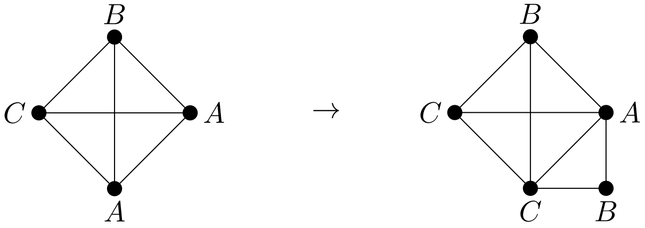


$d(G) - d^c(G)$ for Highly Computable Graphs

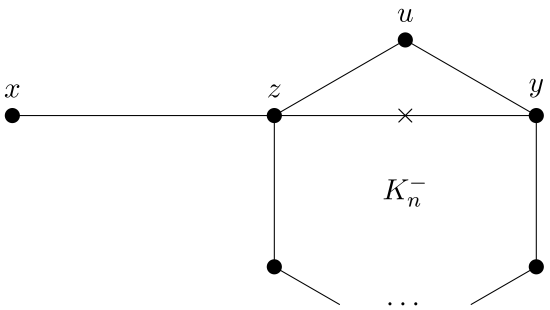
Definition

A graph $G = (V, E)$ is **highly computable** if V and E are computable sets and there is a computable function that, when given $v \in V$, outputs the degree of v (i.e., the number of vertices adjacent to v).

Theorem

For every $n \geq 3$, there is a highly computable graph G such that $d(G) = n$ and $d^c(G) = n - 1$.

The K_n^--Gadget



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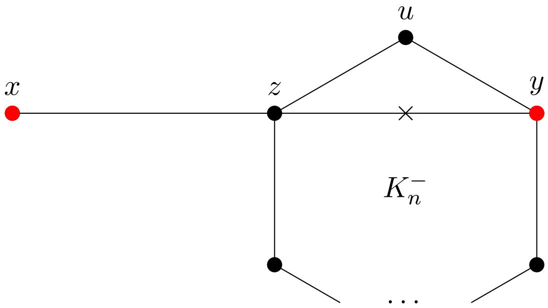

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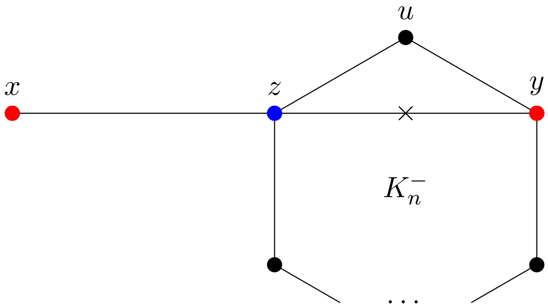
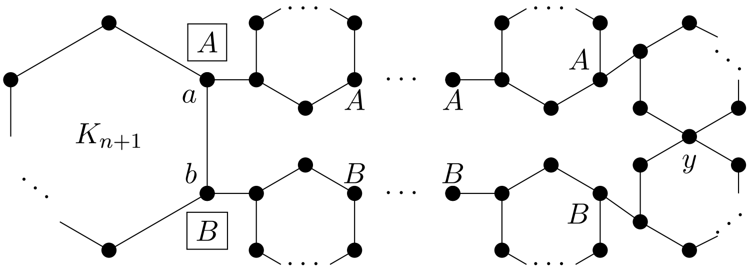


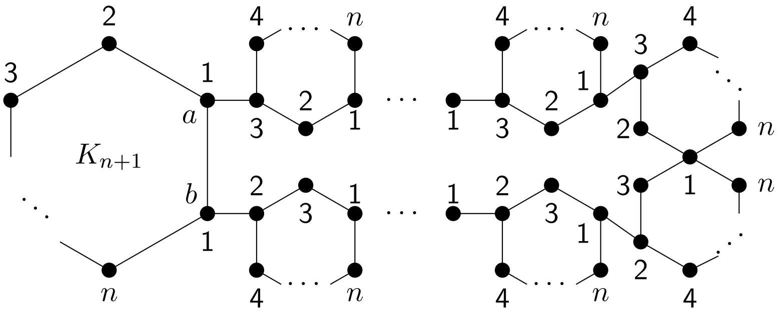
$d^c(G) < n$

Figure : Trapping a purported computable domatic n-partition.



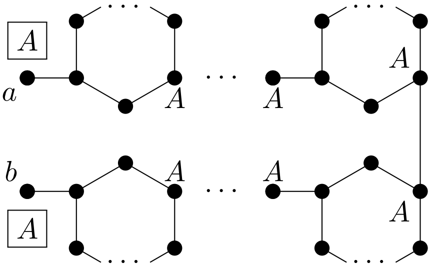
$$d(G) = n$$

Figure : A domatic n-partition of the trap.



An Alternative Trap

Figure : Completing a K_n^- Loop.



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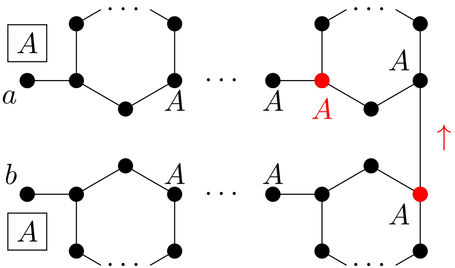


A Graph's Minimal Degree

Definition

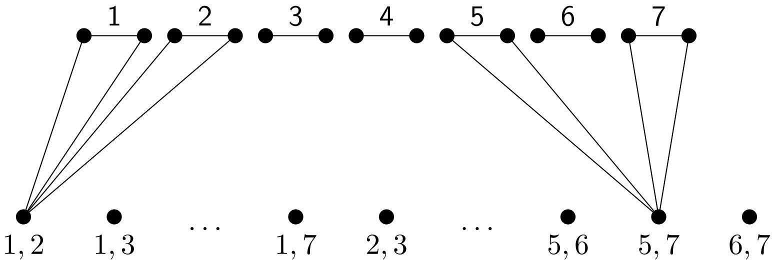
The **minimal degree** of a graph G, denoted by $\delta(G)$, is the minimal degree of the vertices of G.

Theorem (A version of a graph by Zelinka [2])

There exists a computable graph G such that $\delta(G) = 4$, $d(G) = 3$, and $d^c(G) = 2$.

A Zelinka-Type Construction

Figure : Example for $\delta(G) = 4$ and $d(G) = 3$.



The Union of Zelinka's Graph with Another

Theorem

There exists a computable graph G such that $\delta(G) = 4$, $d(G) = 3$, and $d^c(G) = 2$.

Proof.

Let $G = G_1 \cup G_2$, where

- $\delta(G_1) = 4$ and $d(G_1) = 3$ (i.e., G_1 is our Zelinka-type graph);
- G_2 is computable such that $d(G_2) = 5$ (so $\delta(G_2) \geq 4$) and $d^c(G_2) = 2$.

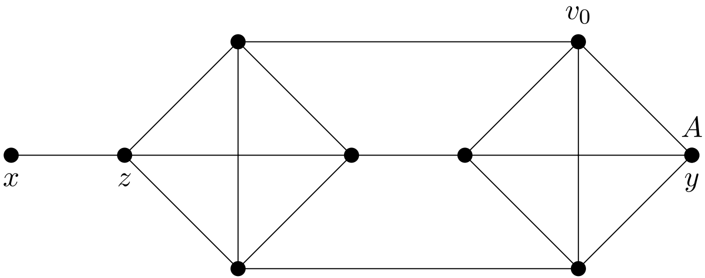
□

Total Domatic Partitions

Definition

For any $n \geq 1$ and any graph $G = (V, E)$, a (computable) partition $p : V \to \{1, \ldots, n\}$ into n colors is a **(computable) total domatic n-partition** if the vertices adjacent to v use up all n colors (i.e., $(\forall v \in V)(\forall i \in \{1, \ldots, n\})(\exists u \in V)[uEv \wedge p(u) = i]$).

The Double K_4-Gadget



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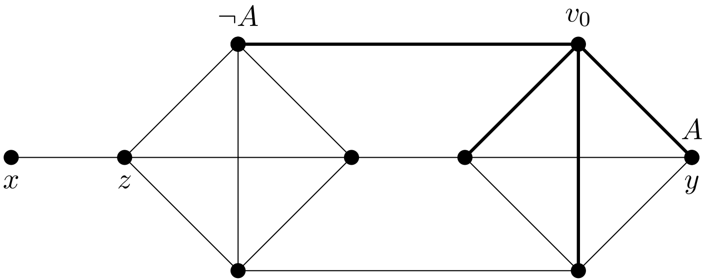

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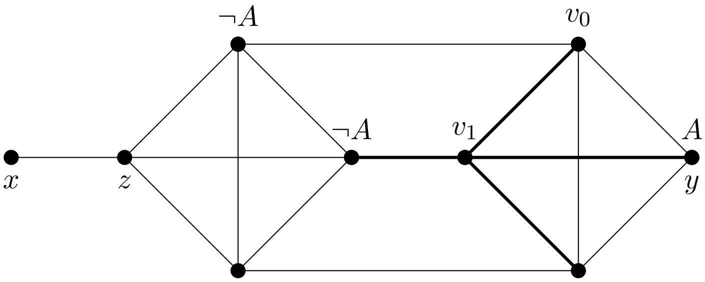

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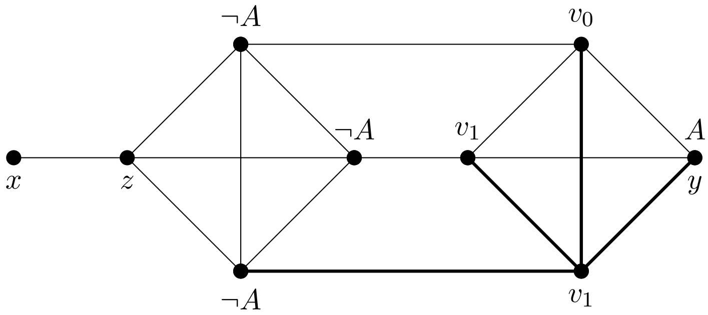

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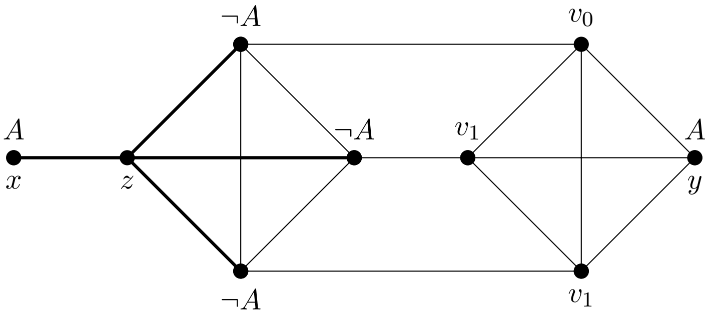

The Regular 3-2 Problem

Definition

For any n, a graph is **n-regular** if the degree of every vertex is n.

Theorem

Every computable 3-regular graph that has a total domatic 3-partition has a computable total domatic 2-partition.

Double Coloring

Vertices on the border of two sets V_0 and V_1 will be doubly colored. Each doubly colored vertex (V_0's Color Choice/V_1's Color Choice) will be resolved by the rules: **If red is present (as either set's choice), choose red. Otherwise, choose blue.**

Figure : Three Cases.

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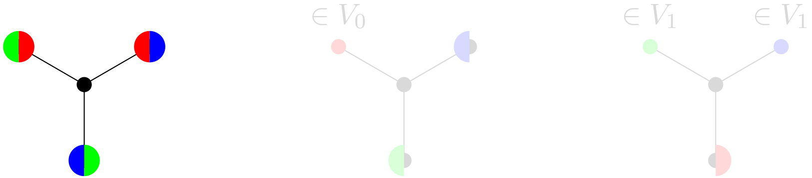

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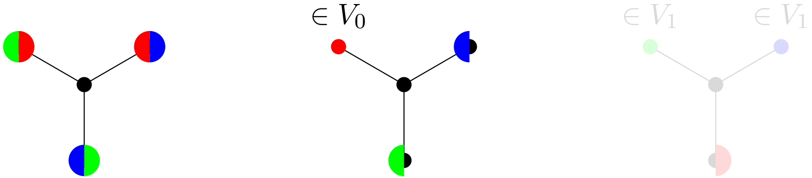

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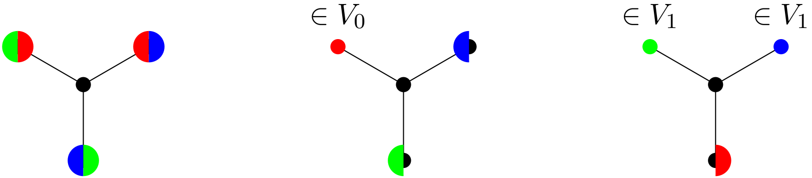

Future Research

Conjecture

Every highly computable graph with a domatic 4-partition has a computable domatic 3-partition.

Questions?

Thank you.

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