Restricting the Turing Degree Spectra of Structures

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$f : (\subseteq \mathbb{N}) \to \mathbb{N}$ is a **partial computable (p.c.) function** if f is an algorithm (or computer program).
Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ list out all p.c. functions.
Let φ_e^B denote the e-th p.c. function with oracle $B \subseteq \mathbb{N}$.

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Fix $A, B \subseteq \mathbb{N}$.

- *A is B-**computable** (or **computed by** B, or **computed from** B), written $A \leq_T B$, if $\chi_A = \varphi_e^B$ for some e.*
- *A, B are **Turing-equivalent**, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.*
- *The **Turing-degree** of A is $\deg(A) = \mathbf{a} =_{df} \{D \mid A \equiv_T D\}$. We write $\mathbf{c} \leq \mathbf{d}$ if $C \leq_T D$ for some $C \in \mathbf{c}$ and $D \in \mathbf{d}$.*

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Example 1 (The Halting Problem)

$K = \emptyset' = \{e \mid \varphi_e(e)\downarrow\}$, where $\downarrow$ means "stops and has an output"
$\mathbf{0}' = \deg(\emptyset')$

Definition 2 (The Jump Operator)

For $X \subseteq \mathbb{N}$, let $X' = \{e \mid \varphi_e^X(e)\downarrow\}$.

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- *The* **degree** *of a structure* $\mathfrak{A}$, *written* $\deg(\mathfrak{A})$, *is the Turing-degree of the (atomic) diagram of* $\mathfrak{A}$:

$$D(\mathfrak{A}) = \{\varphi(\overline{a}) \mid \varphi(\overline{x}) \text{ is atomic or } \neg\text{atomic} \wedge \mathfrak{A} \models \varphi(\overline{a})\}.$$

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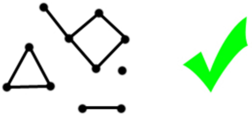
If $\mathfrak{A}$ is a linear order and $S \subseteq \mathbb{N}$ such that $S \in \Sigma_1^{D(\mathfrak{B})'}$ for all $\mathfrak{B} \cong \mathfrak{A}$, then $S \in \Sigma_1^{\emptyset'}$.



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For a finite-component graph $\mathfrak{G}$, let

$$S_{\mathfrak{G}} = \{(C, n) \mid C \text{ is a component of } \mathfrak{G} \text{ occurring at least } n \text{ times}\}.$$

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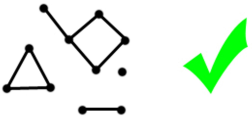

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If $\mathfrak{G}$ is a finite-component graph, $X \subseteq \mathbb{N}$, and $S_{\mathfrak{G}} \in \Sigma_1^X$, then there is a $\hat{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\hat{\mathfrak{G}}) \leq_T X$.

We say that a degree $\mathbf{d}$ *computes* a structure $\mathfrak{A}$ if $\deg(\mathfrak{A}) \leq \mathbf{d}$.

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If $\mathfrak{A}$ is a linear order and $\mathfrak{G}$ is a finite-component graph such that $\mathrm{DgSp}(\mathfrak{A}) \subseteq \mathrm{DgSp}(\mathfrak{G})$, then $\mathbf{0}'$ computes a copy of $\mathfrak{G}$.

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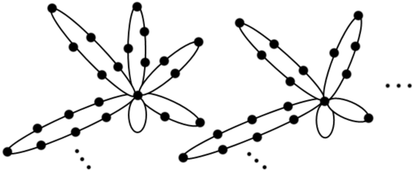
Theorem 3

For each of the following classes $\mathcal{K}$*, there is a linear order* $\mathfrak{A}$ *such that* $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$ *for any* $\mathfrak{B} \in \mathcal{K}$*.*

- *finite-component graphs*
- *equivalence structures*
- *rank-1 torsion-free abelian groups*
- *daisy graphs*

$$\mathcal{S} = \left\{ \{1,2,3,4,6,\ldots\},\ \{0,3,4,6,\ldots\},\ \ldots \right\}$$



$\mathfrak{G}(\mathcal{S})$

Thank you.

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For any degree $\mathbf{d}$, there is a linear order $\mathfrak{L}$ such that $\mathbf{d}$ cannot compute a copy of $\mathfrak{L}$.

Why?

Let $X = D^{(4)}$, $0 \notin X$, for some $D \in \mathbf{d}$. Let $X = \{x_0 < x_1 < \cdots\}$. Define the **shuffle sum** $\sigma(X)$ of X as follows. Let $u_0, u_1, \ldots$ partition $(\mathbb{Q}, <)$ into sets s.t. u_i is dense in $\mathbb{Q}$. Form $\sigma(X)$ by replacing each $q \in u_i$ with x_i many points.

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If $\mathfrak{A}$ is a linear order and $\mathfrak{G}$ is a finite-component graph such that $\mathrm{DgSp}(\mathfrak{A}) \subseteq \mathrm{DgSp}(\mathfrak{G})$, then $\mathbf{0}'$ computes a copy of $\mathfrak{G}$.

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