Can Computers Do Math? An Introduction to Computability Theory and Effective Mathematics

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Joint with Matthew Jura and Oscar Levin.

Machines & Self-Reference

- Consider a machine M that prints out expressions made with five symbols:

$$\sim, \quad P, \quad N, \quad (, \quad \text{and} \quad).$$

- An **expression** is any non-empty finite string of symbols, e.g.,

$$N{\sim}(P, \qquad)P(((((((, \qquad \text{and} \qquad P(N(\sim)).$$

- For an expression X, a **sentence** is any expression of the form:

$$P(X), \quad PN(X), \quad {\sim}P(X), \quad \text{or} \quad {\sim}PN(X).$$

- We interpret the meaning of the symbols as follows.
  P: "is printable"
  $\sim$: "not"
  N: "the **norm** of" E.g., the norm of $P{\sim}$ is $P{\sim}(P{\sim})$.

Telling the Truth

Rule:
The machine M can only print *TRUE* sentences.

Example 1

1. If M prints $P(X)$, then X is printable. So M eventually prints X.
2. If M prints $\sim PN(X)$, then the norm of X, i.e., $X(X)$, is not printable. So M never prints $X(X)$.
3. If M prints X, then M does not necessarily print $P(X)$.

Question:
Can such an M print *ALL* true sentences?

You Can't Handle the Truth!

Can such an M print *ALL* true sentences?

No. The following sentence is true but M will not print it:

$$\sim PN(\sim PN)$$

Notice:

$$\sim PN(\sim PN) \text{ is true} \iff \sim PN(\sim PN) \text{ is not printable}$$

Sets of Natural Numbers

- The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ — countable
- The subsets of $\mathbb{N}$: $\varnothing, \mathbb{N},$

$$\{0\}, \{1\}, \{2\}, \{3\}, \ldots,$$
$$\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}, \ldots,$$
$$\{0,1,2\}, \ldots,$$
$$\vdots$$
$$\{0,2,4,6,8,\ldots\}, \{1,3,5,7,9,\ldots\},$$
$$\{2,3,5,7,11,\ldots\}, \ldots$$
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— uncountably many

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What Is a Computer Program?

Example 2 ($a + b$)

Input: a, b
Want **Output:** $a + b$
Program:

1. $s := a$
2. $i := 0$
3. IF $\{ \quad i \neq b$
   i. $s := s + 1$
   ii. $i := i + 1 \}$
4. Print s

A **(computer) program** is a machine with a finite list of steps (written from a finite alphabet) that takes in a natural number (the **input**), runs the steps on the input, and (if it stops running) prints out a natural number (the **output**).

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Examples of Programs

A computer program is a partial function $f : \mathbb{N} \to \mathbb{N}$ (which may not have an output for some inputs).

Example 3 (Familiar programs)

$a + b$, $a \dot{-} b$, $a > b$, $a \cdot b$, a^b, $a | b$, EVEN(a), ODD(a)

Example 4 (A program where some inputs have no output)

Program Name: EVENstopODDdontstop

Input: a

Program:

1. If a is EVEN, Then Print 0.
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Computable Functions and Sets

- Each program is a finite list of steps. So how many different programs are there?
- So we can number ALL of the programs:

  $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \cdots$

Definition 1

- *We call these φ_e the* **partial computable functions**. *If φ_e is total (i.e., $\mathrm{dom}(\varphi_e) = \mathbb{N}$), we call it a* **computable function**.
- *$A \subseteq \mathbb{N}$ is called a* **computable set** *if*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

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Examples of Computable Sets

Example 5

- $\varnothing$

  because $\chi_\varnothing(x) = \begin{cases} 1 & \text{if } x \in \varnothing \\ 0 & \text{if } x \notin \varnothing \end{cases} = 0 \quad \text{(for all } x\text{)}$

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- A, where A is a finite set
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- Are there any sets that are NOT computable?
- There are only many computable sets, but many subsets of $\mathbb{N}$.
- There are many non-computable sets!
- Can I see one?

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A Not-Computable Set

Example 6

The **Halting Problem**:
$K = \{e : \varphi_e(e)\downarrow\}$

($\varphi_e(e)\downarrow$ means "φ_e on input e stops running")

What does K mean?

- Say EVENstopODDdontstop is φ_{12}, so $e = 12$. Does $\varphi_{12}(12)$ stop or not stop?
- K is "ALL the e such that $\varphi_e(e)$ stops running."

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Why Is $K = \{e : \varphi_e(e)\downarrow\}$ Not Computable?

- Assume K were a computable set. That is, assume $\chi_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$ is a computable function.
- **Claim:** The following function is also computable:

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

Why? **Input:** x

Program:

If $\chi_K(x) = 1$, Then Print $\varphi_x(x) + 1$

If $\chi_K(x) = 0$, Then Print 0

On the other hand: $f \neq \varphi_e$ for any e because:

If $e \in K$, then $f(e) = \varphi_e(e) + 1 \neq \varphi_e(e)$.

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Why? **Input:** x

Program:

If $\chi_K(x) = 1$, Then Print $\varphi_x(x) + 1$

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On the other hand: $f \neq \varphi_e$ for any e because:

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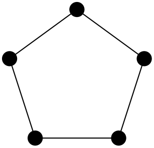
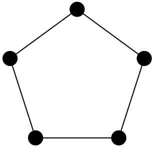
Domatic Partitions

Domatic 2-partition



No domatic 3-partition



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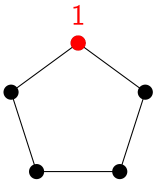

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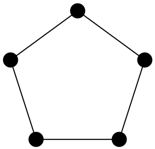

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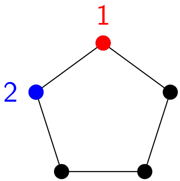

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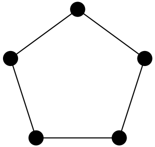

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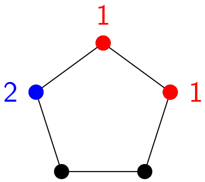

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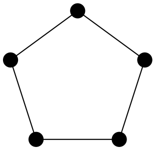

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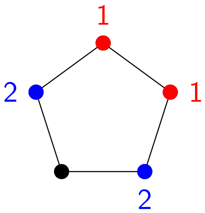

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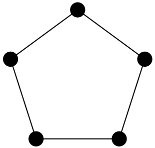

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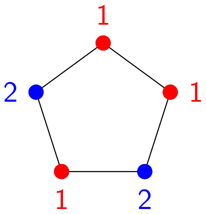

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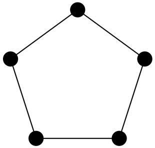

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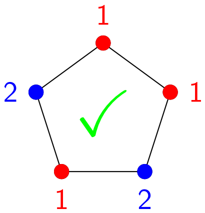

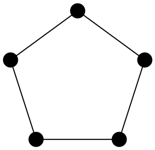

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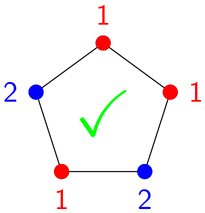

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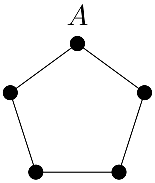

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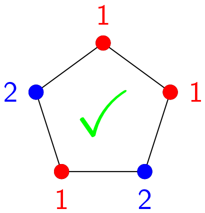

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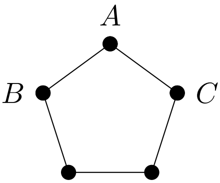

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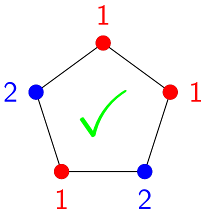

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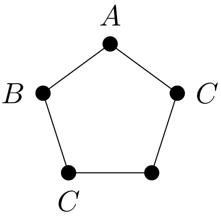

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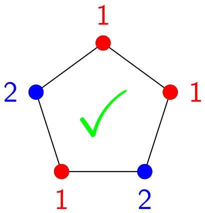

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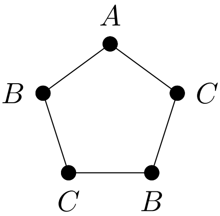

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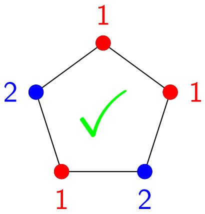

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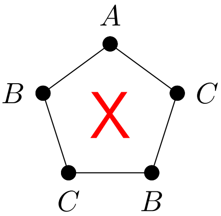

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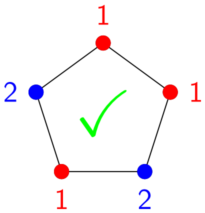

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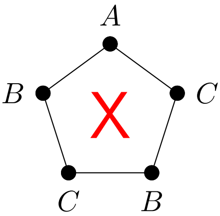

$d(G) = 2$

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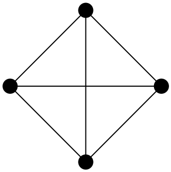
Separating the Domatic Numbers

Example 7

There is a computable graph G such that $d(G) = 3$ but $d^c(G) < 3$.

The gadget of φ_e:



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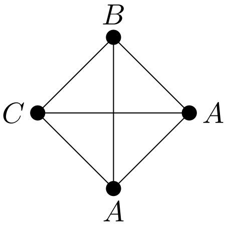

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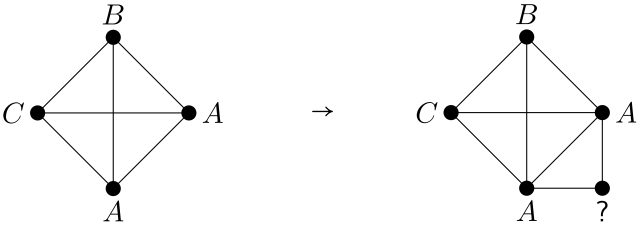

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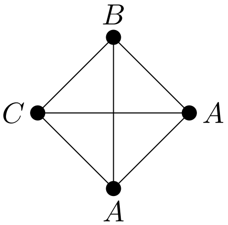

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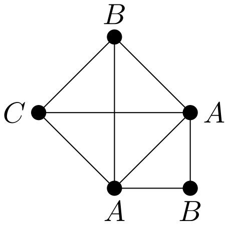

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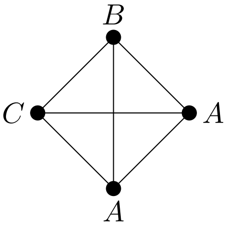

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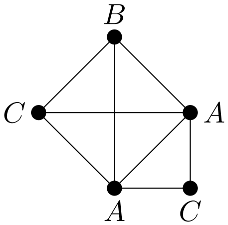

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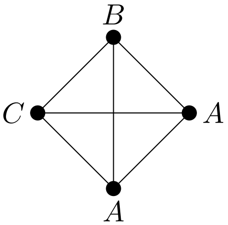

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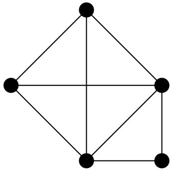

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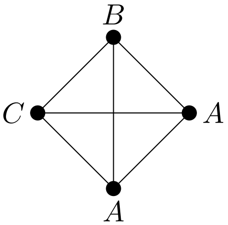

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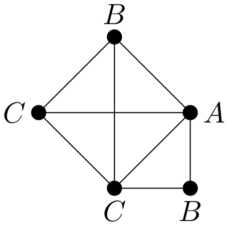

$d(G) - d^c(G)$ for Highly Computable Graphs G

Definition 2

A graph $G = (V, E)$ is **highly computable** *if V and E are computable sets and there is a computable function that, when given $v \in V$, outputs the degree of v (i.e., the number of vertices adjacent to v).*

Theorem 1

For every $n \geq 3$, there is a highly computable graph G such that $d(G) = n$ and $d^c(G) = n - 1$.

The K_n^--Gadget

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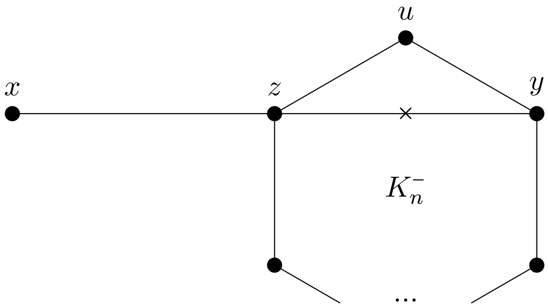

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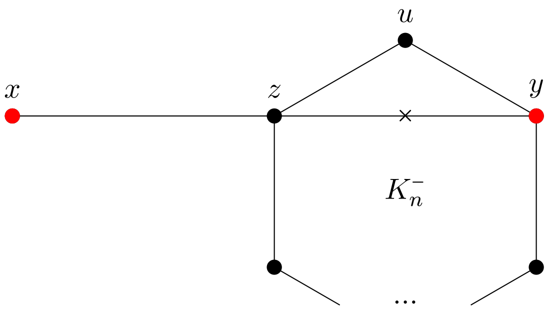

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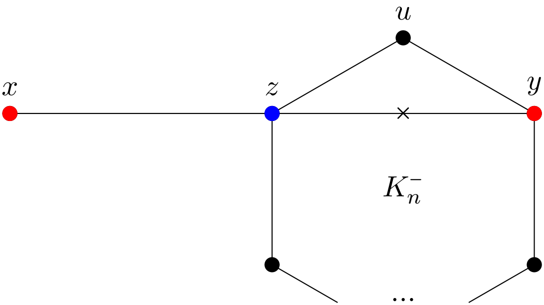

Total Domatic Partitions

Definition 3

1. *For any $n \geq 1$ and any graph $G = (V, E)$, a (computable) partition $p : V \to \{1, \ldots, n\}$ into n colors is a* **(computable) total domatic n-partition** *if the vertices adjacent to v use up all n colors (i.e., $(\forall v \in V)(\forall i \in \{1, \ldots, n\})(\exists u \in V)[uEv \wedge p(u) = i]$).*
2. *The* **(computable) total domatic number** *of a graph G, denoted by $d_t(G)$ (resp., $d_t^c(G)$) is the maximum n such that G has a (computable) total domatic n-partition.*

Theorem 2

For every $n \geq 3$, there is a highly computable graph G such that $d_t(G) = n$ and $d_t^c(G) = n - 1$.

The Double K_4-Gadget

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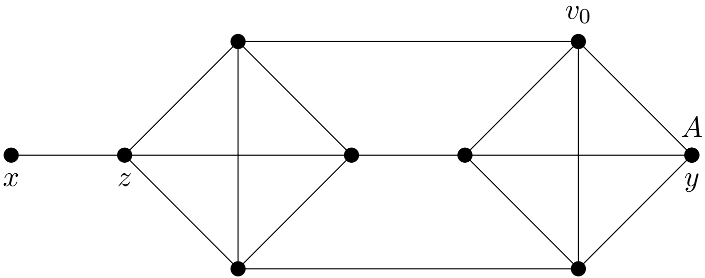

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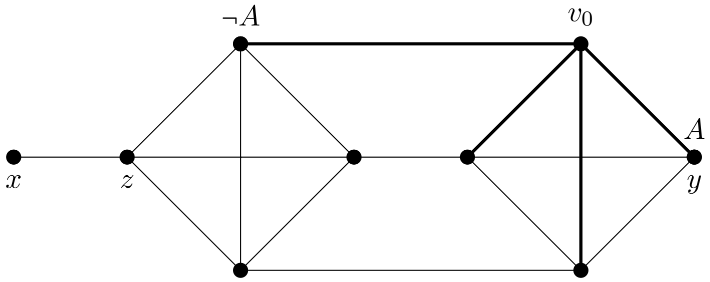

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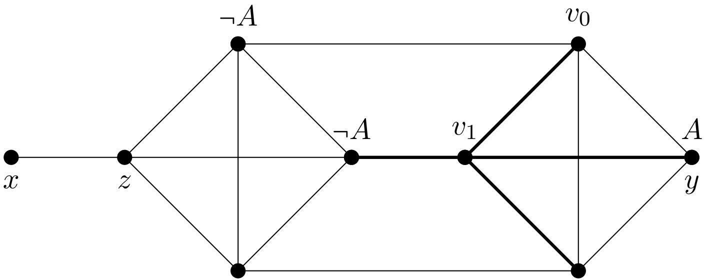

The Double K_4-Gadget

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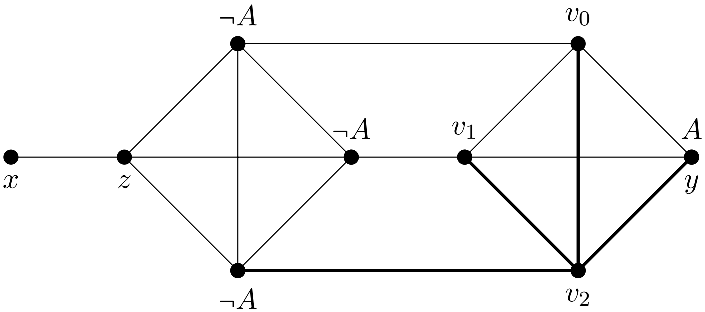

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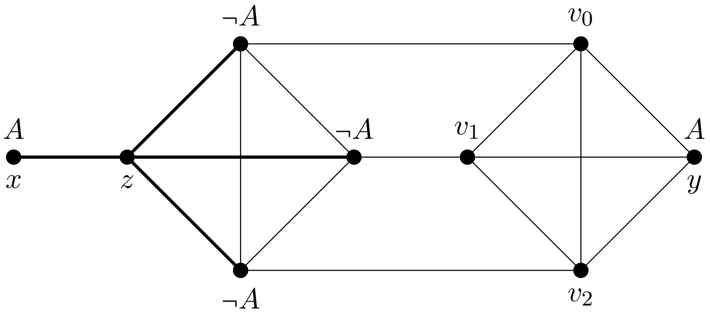

Future Research

Conjecture 1

Every highly computable graph with a domatic 4-partition has a computable domatic 3-partition.

Theorem 3

Let G be a computable k-regular graph with $d(G) = k + 1$. Then G has a computable domatic n-partition for all n satisfying $2^n - 1 \leq k + 1$, in fact, $n^2 \leq k + 1$.

Thank you.

Matthew Jura, Oscar Levin, and Tyler Markkanen.
Domatic partitions of computable graphs.
Arch. Math. Logic, 53(1-2):137–155, 2014.

Robert I. Soare.
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