Cardinality Puzzles

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This is joint with Oscar Levin.

Definition

- A **set** is any collection of items (usually numbers). Each item is called an **element** of the set.
- The **cardinality** of a finite set is the number of elements in the set.

Example

- Let $S = \{0, 17, 8, 9\}$.
- 17 is an element of the set S, which is written as $17 \in S$.
- 5 is not an element of S, and this is written as $5 \notin S$.
- The cardinality of S is denoted by $|S|$, and in this case $|S| = 4$.

Sets are determined by what their elements are. So repetition of elements does not count, and neither does order.

Example

- The set $\{4, 2, 4, 3\}$ is the same set as $\{4, 2, 3\}$.
- The set $\{4, 2, 3\}$ is the same set as $\{2, 3, 4\}$.

Definition

We define a **cardinality puzzle** to be any description of a set that explicitly mentions the cardinality of itself.

Puzzle

Let $A = \{2, |A|\}$. What is the cardinality of the set A?

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Puzzle

What is the cardinality of the set $A = \{2, |A|\}$?

Solution

Notice that A appears to have two elements. That is, $|A| = 2$. But this observation has the following consequence:

$$|A| = 2 \implies A = \{2, 2\} = \{2\} \implies |A| = 1$$

Let's see what this now leads to:

$$|A| = 1 \implies A = \{2, 1\} \implies |A| = 2$$

We are in a vicious cycle, so this puzzle seems to have no solution.

This is similar to the **Liar Paradox**: "This sentence is false." If it is indeed false, then it's true! And if it's true, then it's false again!!

Puzzle

What is the cardinality of the set $A = \{2, 3, |A|\}$?

Solution

It appears to have three elements: $|A| = 3$. But this leads to:

$$|A| = 3 \implies A = \{2, 3, 3\} = \{2, 3\} \implies |A| = 2$$

This in turn leads to:

$$|A| = 2 \implies A = \{2, 3, 2\} = \{2, 3\} \implies |A| = 2$$

That works! So this is not a paradox. It has a solution: the set is $A = \{2, 3\}$ and its cardinality is two.

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No Solutions: Given any natural number $k \geq 1$, the following cardinality puzzle is a paradox:

$$A = \{1, 2, \ldots, k, k+2, |A|\}$$

Example ($k = 1$)

The following cardinality puzzle is a paradox: $A = \{1, 3, |A|\}$.

Notation: For convenience, we'll often use the symbol a instead of $|A|$. So the puzzle shown above can be written as $A = \{1, 3, a\}$.

Here's a cardinality puzzle with two solutions.

Example

$A = \{4, |A|, 2|A|\} = \{4, a, 2a\}$

Once again, it appears to have three elements.

$$|A| = 3 \implies a = 3 \implies A = \{4, 3, 2(3)\} = \{4, 3, 6\} \checkmark$$

Since repetition doesn't count in sets, and since 4 is the only constant element of A, could $|A| = 1$? How about $|A| = 2$?

$$a = 1 \implies A = \{4, 1, 2(1)\} \implies A = \{4, 1, 2\} \text{ X}$$
$$a = 2 \implies A = \{4, 1, 2(2)\} \implies A = \{4, 2, 4\} = \{4, 2\} \checkmark$$

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Each of the following cardinality puzzles has a unique solution.

1. $A = \{a, a+1, a+2\}$
2. $A = \{4, 5, a, a+1, 2a-1\}$

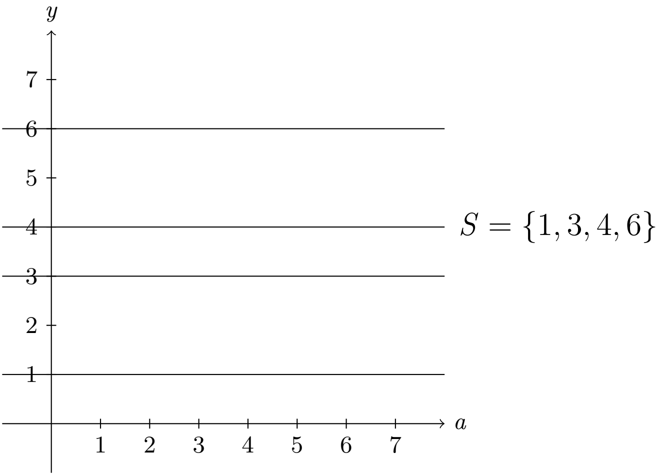
In fact, they both have the same solution: $\{3, 4, 5\}$. But notice that the first has three possible cardinalities (1, 2, or 3), and the second has four (2, 3, 4, or 5). So they have different *complexities*.

Definition

The **complexity** of a cardinality puzzle is the number of its possible cardinalities. Technically, it is the number m of variable elements displayed if there are no constant elements, and $m+1$ otherwise.

Goal: Given any *selfish* set S (meaning that $|S| \in S$) and any natural number m, build a cardinality puzzle of complexity m such that S is its unique solution.

Let's build a cardinality puzzle from the selfish set $S = \{1, 3, 4, 6\}$.



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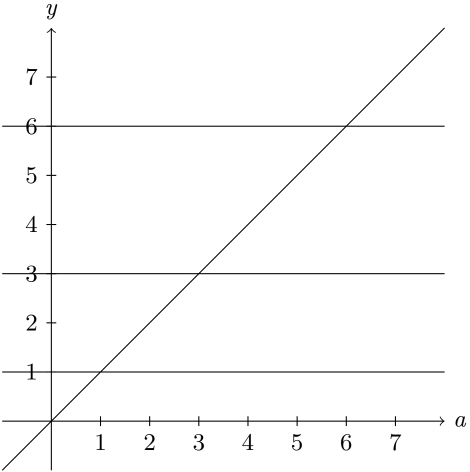

$$\{1, 3, a, 6\}$$

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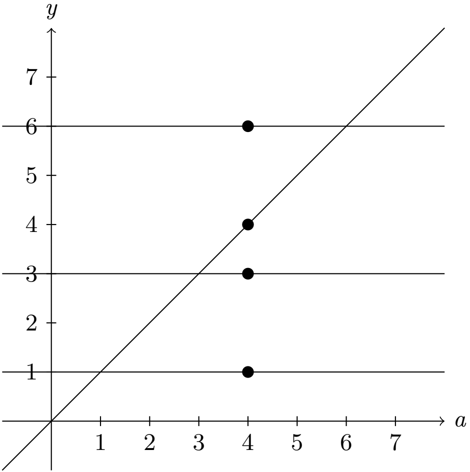

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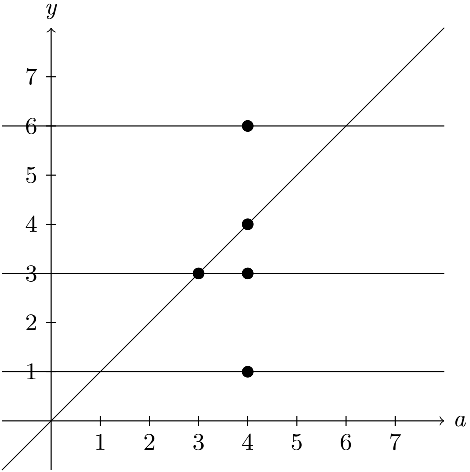

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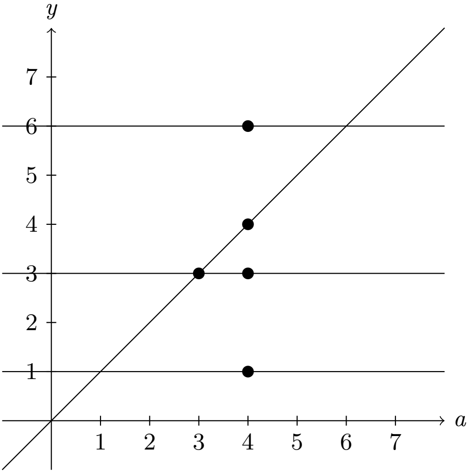

$\{1, 3, a, 6, f(a)\}$

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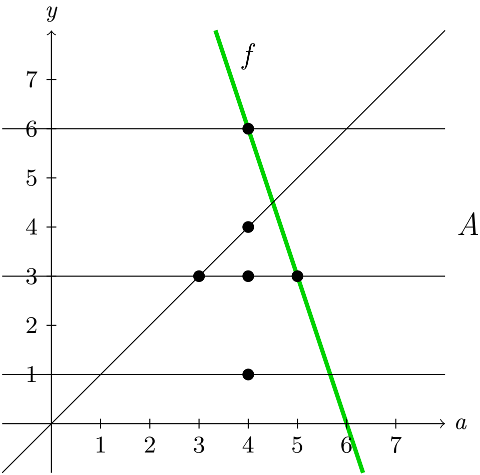

$A = \{1, 3, a, 6, f(a)\}$

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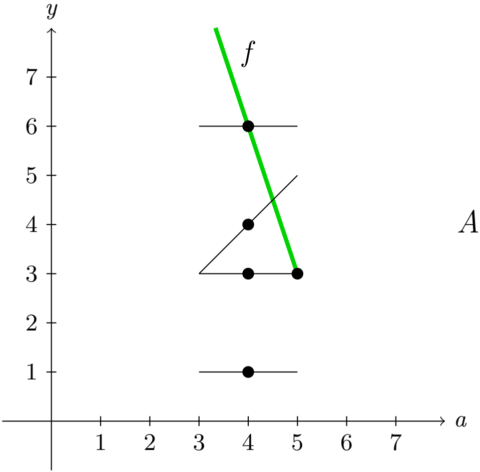

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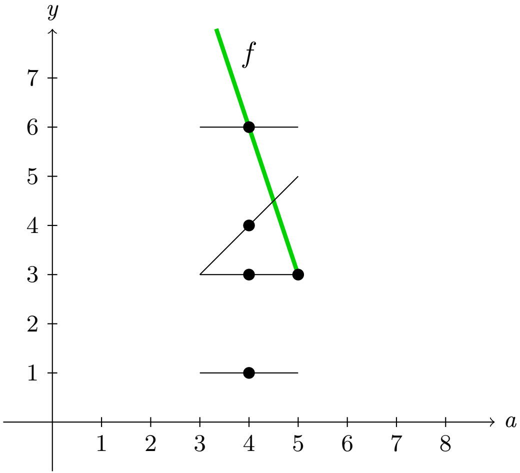
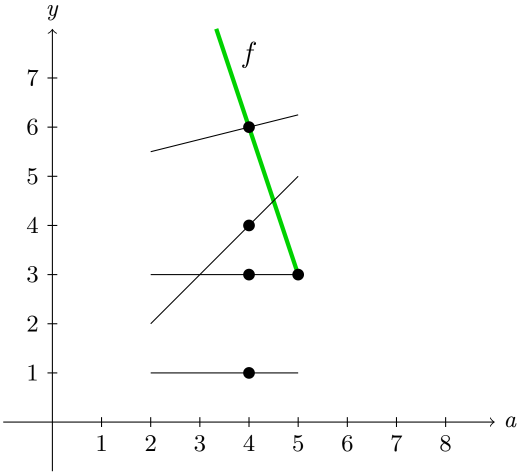


Cardinality Puzzle:

$A = \{1, 3, a, 6, f\}$

Complexity:

3

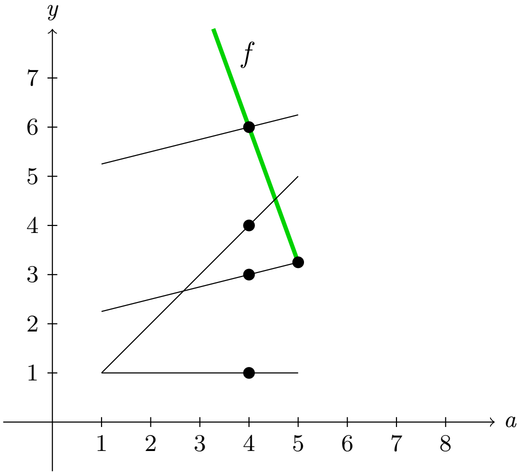


Cardinality Puzzle:

$A = \{1, 3, a, f_6, f\}$

Complexity:

4

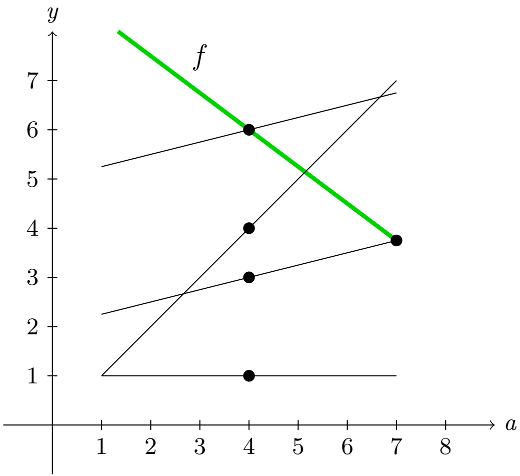


Cardinality Puzzle:

$A = \{1, f_3, a, f_6, f\}$

Complexity:

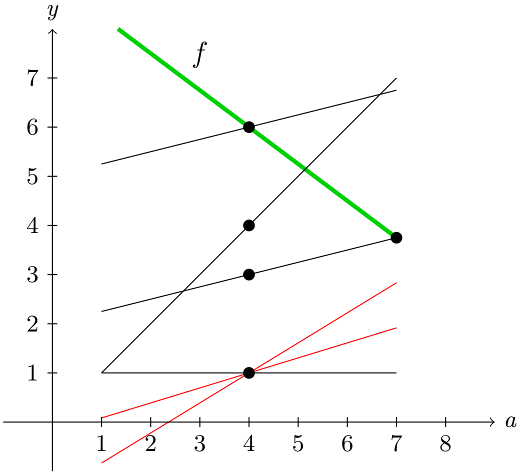
5



Cardinality Puzzle:

$A = \{1, f_3, a, f_6, f, \quad, \quad\}$

Complexity:



Cardinality Puzzle:

$A = \{1, f_3, a, f_6, f, g, h\}$

Complexity:

7

Kight-and-Knave Problems: We encounter some trolls. Each troll is either a knight or a knave. Knights always tell the truth, and knaves always lie. Can we determine which one is which?

Problem (Two Trolls)

Troll 1: We are both knights.
Troll 2: Troll 1 is a knave.

Solution

Assume Troll 1 is a knight. Then Troll 2 is also a knight but lying at the same time. This is impossible!
So Troll 1 is a knave. Thus Troll 2 speaks the truth, meaning that Troll 2 is a knight.

Puzzle (Two Sets)

$A = \{1, a, b\}$ "We are both knights."
$B = \{3, a\}$ "A is a knave."

(where $a = |A|$ and $b = |B|$)

Solution

Assume $a = 3$. Then $b = 2$ (based on the elements of A, which show that $b \neq 1$). But at the same time, $B = \{3, 3\} = \{3\}$. So $b = 1$. This is impossible!
So $a \neq 3$. Thus $b = 2$ (based on the elements of B).
In the end $A = \{1, 2, 2\} = \{1, 2\}$ and $B = \{3, 2\}$.

A is playing the role of Troll 1, and B the role of Troll 2. Indeed, A lied about how many elements it had, and B told the truth.

Three Trolls & Three Sets

T1: Only one of us is a knave. $A = \{1, 3, 5, 6, 7, b, c - 7\}$
T2: No, only one of us is a knight. $B = \{7, 11, a, c\}$
T3: We are all knaves. $C = \{4, 7, 11, 12, \ldots, 16, a, b, c\}$

Solution

First, $c \neq 11$ (T3 must be lying), because if $c = 11$ then $|C| \leq 10$. So $|C| = 9$ or 10, which implies that $a = 7$ or $b = 4$ (T1 or T2 is a knight), or both. Assume $a = 7$. Since c is neither 11 nor 7, $b = |B| \geq 3$. But $b \neq 3$ according to A, so $|B| = b = 4$. Thus $a \neq 7$, contradicting our assumption. So $b = 4$ (T2 was a knight all along), and $a \neq 7$ (T1 is a knave). Notice $a = |A| \geq 6$, so in fact $a = 6$. Plugging this all into C shows us $c = 10$. In the end it's:

Knave,	Knight,	Knave
$\{1, 3, 4, 5, 6, 7\}$,	$\{6, 7, 10, 11\}$,	$\{4, 6, 7, 10, 11, 12, \ldots, 16\}$

Thank you.