

The Computational Strength of Matchings in Countable Graphs

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Abstract

In a 1977 paper, Steffens identified an elegant criterion for determining when a countable graph has a perfect matching. In this paper, we will investigate the proof-theoretic strength of this result and related theorems. We show that a number of natural variants of these theorems are equivalent, or closely related, to the “big five” subsystems of reverse mathematics.

The results of this paper explore the relationship between graph theory and logic by showing the way in which specific changes to a single graph-theoretical principle impact the corresponding proof-theoretical strength. Taken together, the results and questions of this paper suggest that the existence of matchings in countable graphs provides a rich context for understanding reverse mathematics more broadly.

Keywords: computability theory, reverse mathematics, graph matching

2010 MSC: 03D80, 03D99, 03F15, 03F35

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March 9, 2021

1. Introduction

The search for necessary and sufficient conditions for a graph to have a perfect matching has a long and distinguished history, and has resulted in a range of theorems both for bipartite graphs and for graphs in general. For theorems concerning matchings in bipartite graphs, much research has been done to classify their computational and proof-theoretical strength [1, 2, 3, 4].

In this paper, we will turn our attention to classifying the strength of theorems about the existence of matchings in *general* graphs. In particular, we will study a number of related theorems due to Steffens [5], who gave a particularly elegant criterion for the existence of perfect matchings in countable graphs.

1.1. The existence of perfect matchings

A graph $G = (V, E)$ is a set of vertices V together with a set of edges E , each edge a two-element subset of V . Thus, a graph is simple (without multiple edges and without loops) and undirected, but need not be connected or even have any edges at all. A *matching* is a subset $M \subseteq E$ so that no vertex of G is incident to more than one edge of M . Thus, the empty set is also a matching. A matching is *perfect* if every vertex of G is incident to an edge of M .

Now suppose that M is some matching, and that v is a vertex not covered by M . It is natural to ask whether M can be modified to cover v , by starting at v and repeatedly adding and removing edges. To make this precise, consider a matching M and a path $P = (v_i)_{i < k \leq \omega}$, where a *path* P is an injective sequence of vertices such that $\{v_i, v_{i+1}\} \in E$ for each $i + 1 < k$.

A path is called *M -alternating* if the edges $\{v_i, v_{i+1}\}$ alternately lie in M and $E \setminus M$ (the trivial path, consisting of a single vertex, is automatically M -alternating). An M -alternating path P is called *M -augmenting* if it starts at a vertex $s \in V(G) \setminus V(M)$ and either (1) P is infinite, or (2) P terminates in a vertex $v \in V(G) \setminus V(M)$ with $v \neq s$.

Notice that if P is an M -augmenting path, then swapping the membership of those edges in the path results in a strictly larger matching. In other words, the existence of an M -augmenting path allows us to extend the matching M to cover an additional one or two vertices. We often identify P with its constituent edges, so this process of alternating the edges of M which are a

part of P , while keeping every edge of M that doesn't touch P , is the same as taking the symmetric difference $M \Delta P$ of M and the edges of P . It is easy to see that this is still a matching by considering the edges at each vertex, and now the vertex s is covered by the new matching (as is v in the case that the M -augmenting path terminates).

It is natural to ask whether there is a connection between the existence of perfect matchings and the existence of augmenting paths for imperfect matchings. For countable graphs, this question was answered by Steffens [5] using the following terminology.¹

Definition 1.1. A countable graph G is said to satisfy *condition (A)* if for every matching M and for every vertex $s \in V(G) \setminus V(M)$ there exists an M -augmenting path which starts at s .

Theorem 1.2 (Steffens [5]). *A countable graph G has a perfect matching if and only if G satisfies condition (A).*

The implication in one direction is straightforward.

Proposition 1.3. *If a countable graph G has a perfect matching, then G satisfies condition (A). Furthermore, this holds over RCA_0 .*

Proof. Fix a graph G and a perfect matching N . For any imperfect matching M and any $s \notin V(M)$, comparing M with N allows us to define an M -augmenting path $(v_i)_{i < k \leq \omega}$, as required by the statement of condition (A). To see why, let $v_0 = s$ and v_1 be the neighbor of s such that $\{s, v_1\} \in N$. If $v_1 \notin V(M)$, then $P = (s, v_1)$ is the desired M -augmenting path. Otherwise, if $v_1 \in V(M)$, there is a v_2 such that $\{v_1, v_2\} \in M$. But then, because N matched v_1 to s and because $v_2 \in V(N)$, there must be a distinct v_3 such that $\{v_2, v_3\} \in N$. Continuing like this, either we end with v_k outside of $V(M)$ or we never end. In either case, the path is M -augmenting. Furthermore, this path is clearly Δ_1^0 in M and N , so the path exists by RCA_0 . \square

It follows that the strength of Theorem 1.2 is contained in the implication “If condition (A) holds of a graph G , then there is a perfect matching of G .” We will call this direction of the biconditional the *Perfect Matching Theorem*, or PM.

¹For uncountable graphs G , there is a different condition classifying the G which have a perfect matching, but that condition is more complicated. See Aharoni's results of [6].

Given a matching M , the corresponding vertex set $V(M)$ is called the *support* of M . Steffens points out that his proof of Theorem 1.2 proves an apparently stronger result: even graphs that do not satisfy condition (A) must have a matching of maximal support.

Theorem 1.4 (Steffens [5]). *For each countable graph $G = (V, E)$ there is a maximal subset $V' \subseteq V$ which has a perfect matching.*

We will call Theorem 1.4 the *Maximal Matching Theorem*, or MM. The connection between the main theorems is straightforward, but illuminating.

Proposition 1.5. *Theorem 1.4 implies Theorem 1.2 over RCA_0 .*

Proof. Consider any graph G . By Theorem 1.4, G has a matching M of maximal support. Suppose also that G satisfies condition (A). Then if there is any $v \notin V(M)$, there is an M -augmenting path P starting at that v . But recall from above that because P is an M -augmenting path, then $M \Delta P$ is a matching of G that covers more vertices than M . This contradicts the maximality of M . Furthermore, it is clear that $M \Delta P$ is Δ_1^0 in M and P , so the proof goes through in RCA_0 . \square

The proofs of Theorems 1.2 and 1.4 revolve around a special kind of matching, which will be discussed in more depth in Section 4.

Definition 1.6. A matching M is *independent* if the only M -augmenting paths are length-1 paths that begin and end at vertices outside of M .

Although it is also true that every graph has an edge maximal matching, edge maximal matchings are trivial to construct, and so are not of interest to this paper.

1.2. Classifying computational strength

We will analyze several versions of PM and MM (Theorems 1.2 and 1.4) from the viewpoint of computability theory and reverse mathematics. Indeed, this paper can be seen as a “case study” in how varying specific graph-theoretic features of a mathematical principle directly impacts the logical strength of that principle.

In the next section, Figure 1 summarizes our results showing that versions of Theorems 1.2 and 1.4, obtained by restricting them to different classes of graphs, are either equivalent or closely related to the standard subsystems of second order arithmetic.

This work continues several important veins of research into the reverse mathematics of principles in infinite graph theory. In particular, the current paper is closely related to the reverse mathematics of the König duality theorem for countable bipartite graphs, written CKDT, which relates the existence of matchings to vertex covers of bipartite graphs. This theorem has been shown to be equivalent to ATR_0 . A proof in $\Pi_1^1\text{-CA}_0$ and a reversal to ATR_0 were provided by Aharoni, Magidor, and Shore [1], and Simpson [3] later showed that this theorem is provable in ATR_0 . Continuing this line of work, Shafer [2] studied Menger’s theorem for countable webs, which is a sort of generalization of CKDT. Shafer, in [2], showed that Menger’s theorem is provable in $\Pi_1^1\text{-CA}_0$, and defined an extended version that is equivalent to $\Pi_1^1\text{-CA}_0$. Later, Towsner [4] refined this result to show that Menger’s theorem for countable bipartite graphs is actually provable in the system that he defines in [4] and calls TLPP_0 (which stands for “transfinite leftmost path principle”), which lies strictly between ATR_0 and $\Pi_1^1\text{-CA}_0$.

The work in the current paper continues this trend of increasing complexity. It is not difficult to see that the statement itself of PM is significantly more complex than either CKDT or Menger’s theorem for countable webs. In particular, those two theorems are defined by Π_2^1 sentences, while PM is equivalent to a Π_3^1 sentence. In other words, PM is particularly interesting since it has a similar flavor to these other principles, but with a significantly higher sentence complexity.

In other ways, this paper builds on research such as that of Hirst [7], [8] who studied a number of variants of Hall’s theorem concerning matchings of bipartite graphs. By considering different necessary and sufficient conditions for a graph to have a perfect matching, Hirst obtained principles equivalent to systems around the levels of ACA_0 and WKL_0 . The current paper continues and extends this line of research, supporting and expanding our understanding of the relationship between specific graph-theoretic features and reverse mathematics.

We assume that the reader is familiar with computability theory and reverse mathematics, including the “big five” subsystems of second order arithmetic: RCA_0 , ACA_0 , WKL_0 , ATR_0 , and $\Pi_1^1\text{-CA}_0$. Other subsystems will be defined when they are introduced. For additional background on computability theory, see [9] or [10]. For additional background on reverse mathematics, see [11].

2. Summary of results

We will study the existence of three main types of matchings in countable graphs: perfect matchings, matchings of maximal support, and maximal independent matchings. More precisely, we will use the following definitions, formalized in second order arithmetic.

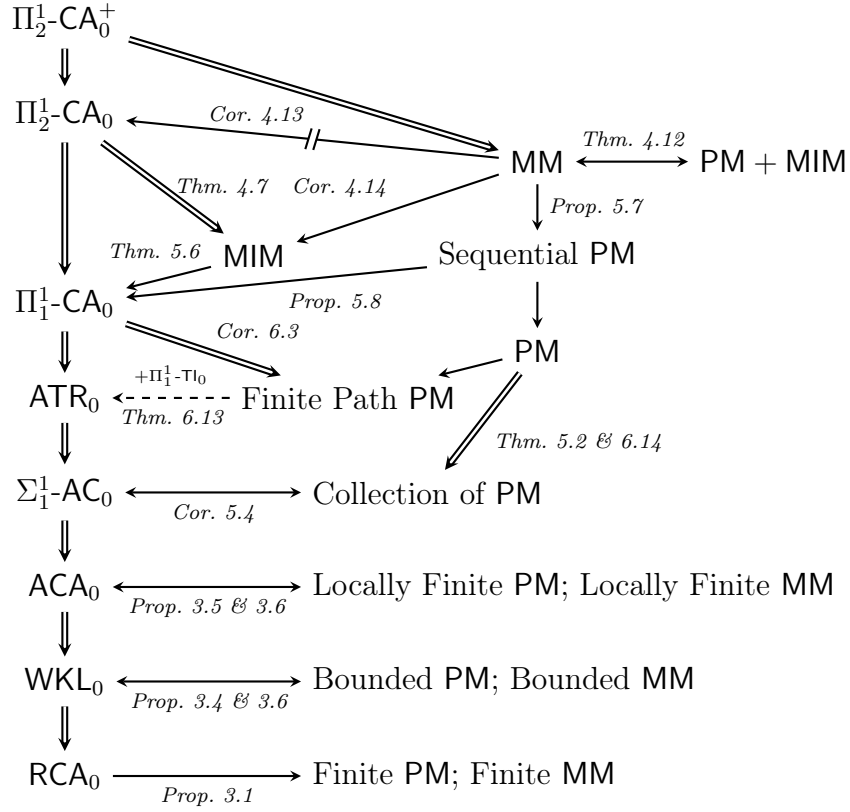


Figure 1: Summary of results

Statement 2.1. The following statements can be formalized in RCA_0 .

1. The Perfect Matching Theorem, or PM , denotes the statement, *if G is a graph satisfying condition (A), then there is a perfect matching M of G .*
(We also study the natural restriction of PM to specific classes of graphs, including finite graphs, bounded graphs, and locally finite graphs.)

2. The Maximal Matching Theorem, or MM, denotes the statement, *for any graph G , there is a matching M of G so that $V(M)$ is not properly contained in the support of any other matching.*
3. Maximal Independent Matching, or MIM, denotes the statement, *for any graph G , there is an independent matching M of G so that $V(M)$ is not properly contained in the support of any other independent matching.*
4. Collection of PM denotes the statement, *if $X = \langle G_1, G_2, \dots \rangle$ codes a sequence of disjoint graphs, and if each G_i has a perfect matching M_i , then there is a perfect matching M for the graph $G = \bigcup_i G_i$.*
5. Sequential PM denotes the statement, *if $X = \langle G_1, G_2, \dots \rangle$ codes a sequence of disjoint graphs, then PM holds of each G_i .*

The majority of the results of this paper are summarized in Figure 1, which shows relationships between these principles and standard subsystems of second order arithmetic. Solid arrows indicate a proof over RCA_0 , while the dashed arrow indicates a reversal over the specified base system. When a strict implication is known to hold, we use a double arrow. A non-redundant non-implication is indicated using a slash through an arrow.

During the course of the paper, we will study a number of versions of PM obtained by restricting the statement of PM to a specific class of graphs. In Section 3, we study locally finite graphs. There, Locally Finite PM denotes the restriction of PM to graphs in which each vertex has finitely many neighbors. We show that Locally Finite PM is equivalent to ACA_0 over RCA_0 . Similarly, Bounded PM denotes the restriction of PM to only those locally finite graphs where there exists a function bounding the neighborhood relation. In other words, Bounded PM is the restriction of PM to the reverse mathematical analogue of highly computable graphs. We show that Bounded PM is equivalent to WKL_0 over RCA_0 . These results illustrate and support the standard reverse mathematical intuitions concerning the combinatorial content of ACA_0 and WKL_0 .

The picture becomes more interesting in Sections 4, 5, and 6 when considering graphs that are not locally finite. The principles concerning maximal matchings, MM and MIM, are both above $\Pi_1^1\text{-CA}_0$, while MIM is strictly below $\Pi_2^1\text{-CA}_0$, and MM is below $\Pi_2^1\text{-CA}_0^+$, but cannot imply $\Pi_2^1\text{-CA}_0$. The strength of PM falls somewhere between MM and Finite Path PM, where Finite Path PM denotes the restriction of PM to graphs with no infinite paths (recall that a path cannot visit any vertex twice). We show that

Finite Path PM implies ATR_0 over $\Pi_1^1\text{-TI}_0$.

To see why this base system is so unusual, recall that the hypothesis, “ G satisfies condition (A),” is Π_2^1 with parameter G , while the conclusion is Σ_1^1 (also with parameter G). In other words, the hypothesis of PM is more complex than its conclusion. This is in stark contrast with theorems such as König’s Lemma, whose hypothesis is a Π_1^0 sentence (the existence of arbitrarily long finite paths) and whose conclusion is a Σ_1^1 sentence (the existence of an infinite path). When the conclusion is of higher complexity than the hypothesis, it is generally straightforward to verify the properties of any attempted coding. In PM, on the other hand, the hypothesis is more complex than the conclusion. Thus, any attempt to code into PM requires proving a Π_2^1 property in order to draw a Σ_1^1 conclusion. This discussion suggests one of several novel challenges introduced by the complexity of condition (A), which will be a common theme throughout this paper.

3. Matchings for locally finite graphs

We begin by considering restrictions of PM to the simpler case of locally finite graphs.

It is easy to see that PM for finite graphs is provable in RCA_0 .

Proposition 3.1. *RCA_0 proves PM for finite graphs. That is, RCA_0 implies that every finite graph satisfying condition (A) has a perfect matching.*

Proof. Any finite graph G has only finitely many matchings. Because being a finite matching is a Δ_1^0 property, there is a maximal such matching by Σ_1^0 induction. But since the graph satisfies condition (A), this matching of maximal support must cover all vertices, as discussed in the proof of Proposition 1.5. \square

The case of PM for locally finite graphs is more interesting. Steffens [5] points out that this case can be proved directly using the Rado Selection Lemma for such graphs. That proof can be formalized in ACA_0 using the version of Rado’s theorem in Theorem III.7.8 of [11]. To consider the additional case of bounded graphs, and to lay a foundation for ideas that will be important later in this paper, we follow a more direct approach.

In particular, the proof below highlights the roles of both condition (A) and the additional assumption that G is locally finite.

Definition 3.2. A locally finite graph is *bounded* provided there is a function $h : V \rightarrow \mathbb{N}$ such that for all $x, y \in V$, if $\{x, y\} \in E$ then $h(x) \geq y$.

Proposition 3.3. ACA_0 proves PM for locally finite graphs, and WKL_0 proves PM for bounded graphs.

Proof. Suppose G is a locally finite graph with vertex set \mathbb{N} , and which satisfies condition (A). Define a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ by letting $\langle a_0, \dots, a_n \rangle \in T$ if and only if $\{\{0, a_0\}, \dots, \{n, a_n\}\}$ is a matching of $\{0, \dots, n\} \cup \{a_0, \dots, a_n\}$.

Note that any infinite path $\langle a_i : i \in \mathbb{N} \rangle$ through T will correspond exactly to a set of edges $\{i, a_i\}$ that define a perfect matching of $V(G) = \mathbb{N}$. It is clear that T is definable in RCA_0 , and that T is really is a tree since the matching that witnesses the inclusion of $\tau \in T$ also witnesses the inclusion of any prefix of τ .

Note also that for any string $\langle a_0, \dots, a_n \rangle \in T$, $\{i, a_i\}$ must be an edge in G . Because G is locally finite, T will be finitely branching, and in the case that G is bounded, T will also be bounded.

Because G satisfies condition (A), each imperfect matching of G can be extended to one of greater support. The key use of this property comes in the proof that T is infinite. It suffices to show that for each n , there are some vertices v_0, \dots, v_n and some finite matching $I_n = \{\{0, v_0\}, \dots, \{n, v_n\}\}$ of the vertices $F_n = \{0, \dots, n\} \cup \{v_0, \dots, v_n\}$. We prove this using Σ_1^0 induction.

For $n = 0$, by condition (A) applied to the empty matching, there is an \emptyset -augmenting path starting at vertex 0, which just consists of some neighbor of 0, call it v_0 . Then $F_0 = \{0, v_0\}$ has matching $I_0 = \{\{0, v_0\}\}$.

Now suppose $I_n = \{\{0, v_0\}, \{1, v_1\}, \dots, \{n, v_n\}\}$ is a matching of

$$F_n = \{0, 1, \dots, n\} \cup \{v_0, v_1, \dots, v_n\}.$$

If $n+1 \in F_n$, then we are done (let v_{n+1} be the vertex to which $n+1$ is already matched). Otherwise, by condition (A), there is an I_n -augmenting path P_n beginning at $n+1$. Because $V(I_n)$ is finite, the path P_n must be finite, and thus end in some vertex $v_{n+1} \notin F_n \cup \{n+1\}$. Set $F_{n+1} = F_n \cup \{n+1, v_{n+1}\}$, and recall that $P_n \Delta I_n$ is a matching I_{n+1} of F_{n+1} . In this case, note that I_{n+1} will *not* extend I_n , but *will* extend some matching on the tree.

It follows that T is an infinite, finitely branching tree. By ACA_0 there is an infinite path P . In the case that G is bounded then T is also bounded, so the existence of this infinite path P follows from WKL_0 instead.

All that remains is to verify that the set of edges $M = \{\{i, P(i)\} : i \in \mathbb{N}\}$ defines a perfect matching of G . Suppose that M is not a matching. This

means some vertex in G is incident to at least two edges of M . Consider the three vertices incident to these two edges. Now restrict T to the level of T containing all three vertices. This produces a string in T that does not define a matching, contradicting our definition of T . Clearly $V(M) = \mathbb{N}$, so the matching is perfect. Thus M is a perfect matching of G . \square

The upper bounds given above are optimal.

Proposition 3.4. *PM for bounded graphs is equivalent to WKL_0 over RCA_0 .*

Proof. The forward implication is proved in Proposition 3.3. For the reversal, we prove WKL_0 via Σ_1^0 separation, which is sufficient by Lemma IV.4.4 in [11]. Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be one-to-one functions satisfying $\forall i, j [f(i) \neq g(j)]$. We will define a set X such that $\forall m [f(m) \in X \wedge g(m) \notin X]$.

For each $n \in \mathbb{N}$, build a disjoint path of odd length (number of edges), in a way that keeps track of the “center” edge (for example, by using the evens as the endpoints of the center edges and odds for all other vertices). At each stage of the construction, add edges to both ends of each path unless n enters the range of f or g . In this case, stop building the path after first ensuring that the length is either 1 (mod 4) if n is in the range of f , or 3 (mod 4) if n is in the range of g . If n never appears in the range of either function, build the path forever.

The resulting graph will consist of infinitely many disjoint paths, each either a two-way infinite path or else a finite path of odd length. Such a graph satisfies condition (A): given a matching M and a vertex $v \notin V(M)$, follow the one or two paths leading away from v for as long as they are M -alternating. If v is on the end of one path, there is one alternating path leading away from it. In this case, if the alternating path were not augmenting, then it would end in a vertex matched by M . Similarly, if v was in the middle of a path, then the only way neither path leading away from it would be augmenting is if both paths terminate in a vertex matched by M . In both these cases, the path would be even, a contradiction.

Moreover, the graph is bounded (define h using the effective construction of G). Therefore by PM for bounded graphs, there exists a perfect matching of G . Define the set X to be the set of n such that the “center” edge in the path for n is in the matching. If n is in the range of f , then the path for n had length 1 (mod 4), so the only matching includes the center edge, giving $n \in X$. On the other hand, if n is in the range of g then the path is length 3 (mod 4), so the center edge is not in the matching, giving $n \notin X$. For any n

not in either range, the center edge might or might not be in the matching, which is fine. Thus X is a separating set as required. \square

The full locally finite case was previously shown equivalent to ACA_0 by Sakakibara [12]. We include a proof for completeness.

Proposition 3.5 (Sakakibara [12]). *PM for locally finite graphs is equivalent to ACA_0 over RCA_0 .*

Proof. The forward implication is proved in Proposition 3.3. For the reversal, let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function. We want to show that the range of f exists.

Build a graph G consisting of infinitely many disjoint paths, either of length 1 (edge) or 3. Specifically, for each n , put an edge $\{4n, 4n + 2\}$ in the graph. Whenever a number n enters the range of f , add edges $\{j, 4n\}$ and $\{4n + 2, k\}$, where j and k are the least unused odd natural numbers.

Since all vertices of G are part of either a length-1 or length-3 path, G satisfies condition (A): for any matching M and vertex $v_0 \notin V(M)$, if v_0 is even, then it must be adjacent to an unmatched vertex we can search for and find. If v_0 is odd, then we can search for its even neighbor, which if matched will start a length-3 alternating path. Moreover, G is clearly locally finite. Thus there is a perfect matching M of G . From this matching, define the range of f as those n for which $\{4n, 4n + 2\} \notin M$. \square

Recall that MM implies PM over RCA_0 , and that MM is a corollary to Steffens' *proof* of PM. When restricting to locally finite graphs, we will show that these principles are in fact equivalent.

To show that Locally Finite PM implies Locally Finite MM, we will use the fact that Locally Finite PM implies ACA_0 , and similarly for Bounded PM and WKL_0 . In other words, we prove that every locally finite graph has a maximal matching in ACA_0 , and that this holds in WKL_0 for bounded graphs.

Proposition 3.6. *Over RCA_0 , MM for locally finite graphs is equivalent to Locally Finite PM, and MM for bounded graphs is equivalent to Bounded PM.*

Proof. The proof of Proposition 1.5 shows that any maximal matching for a graph satisfying condition (A) is perfect, so the existence of maximal matchings for locally finite (or bounded) graphs implies PM for these graphs.

For the other direction, we will show that ACA_0 proves that every locally finite graph has a maximal matching, and WKL_0 proves that every bounded

graph has a maximal matching. The argument is similar to the proof in Proposition 3.3, except now we will skip vertices for which there is no way to extend the matching.

First, define a sequence $\langle b_i \rangle_{i < k \leq \omega}$ recursively as follows. Let b_0 be the least non-isolated vertex of the graph. For $n > 0$, let b_n be the least vertex for which there is a matching in G that covers $\{b_0, \dots, b_n\}$. The sequence is strictly increasing, and definable over ACA_0 in general, and in fact RCA_0 for bounded graphs. If the sequence $\langle b_i \rangle$ is finite, then we are done, since any matching that witnesses the last term in the sequence will be a maximal matching. In the case where the sequence is infinite, proceed as follows.

Define a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ by putting $\langle a_0, a_1, \dots, a_n \rangle \in T$ iff $\{\{b_i, a_i\} : i \leq n\}$ is a matching of $\{b_0, \dots, b_n\} \cup \{a_0, \dots, a_n\}$. From the definition of $\langle b_i \rangle$, for each i there is a string of length i which will be on this tree. Therefore, the tree must be infinite. For locally finite graphs, the tree is locally finite and definable over ACA_0 , thus in ACA_0 the tree has a path. For bounded graphs the tree is bounded and definable over RCA_0 , so has a path in WKL_0 .

We claim that any path through the tree gives us a maximal matching M . Suppose there was a matching N with $V(N) \supset V(M)$. Let v be the least vertex in $V(N) \setminus V(M)$, and as such $v \neq b_i$ for any $i \in \mathbb{N}$. Let j be greatest such that $b_j < v$. Then restricting N to its edges covering $\{b_0, \dots, b_j, v\}$ gives a finite matching that covers these vertices, which would have put $v = b_{j+1}$, a contradiction. \square

4. Finding matchings in general

The proofs in Section 3 made an essential use of the assumption that G was locally finite, as König's lemma is false for infinitely branching trees.

One naïve approach to proving PM in the general case would be to iteratively use augmenting paths to grow a matching that will cover an increasing number of vertices of the graph. Unfortunately, there are graphs where this repeated augmentation results in a vertex such that the edge that covers it changes infinitely often, and consequently, that vertex will not be covered in the limit.

To avoid this obstacle, Steffens' proof in [5] centers around building matchings that are stable under augmentation. An M -augmenting path is *proper* provided it passes through an edge of M . It is easy to see that an M -augmenting path P is proper if and only if using P to augment M flips an edge in M .

Thus the matchings stable under augmentation are the ones with *no* proper augmenting paths (that is, where augmenting M by any P is equivalent to adding the single edge of P to M).

Definition 4.1 (Steffens [5]). A *matching M is independent* if there is no proper M -augmenting path starting at a vertex $s \in V(G) \setminus V(M)$.

A subgraph G' of G is *independent* if G' has a perfect matching and if every perfect matching of G' is independent as a matching of G (not of G').

Note that the definition of the independence of M is a Π_1^1 sentence with parameter G . These definitions are closely related to each other: it is provable in RCA_0 that a subgraph G' of G is independent if and only if there is at least one independent perfect matching of G' (Lemma 4.6).

Steffens' proof centers around two key insights. First, Steffens showed that chains of independent subgraphs of increasing support can be combined into a single *maximal independent* subgraph of the union. On its own, this does not prove PM or MM; it is conceivable that some maximal independent matching might be contained inside a chain of larger, *non-independent* matchings.

Second, Steffens proves a sequence of lemmas that, together, allow one to extend any imperfect maximal independent matching to cover a single unmatched vertex, while also ensuring that the complement of (the subgraph induced by) this new matching still satisfies condition (A). By iterating this construction (to cover all unmatched vertices), a perfect matching can then be obtained.

Some of Steffens' lemmas are technical and go beyond the scope of this paper. However, we give a brief introduction to the results most important for our arguments. We also include some details to give the reader a sense of the underlying graph theory. The reader should refer to [5] for a complete picture.

To begin, note that the union of two independent subgraphs is *not* always independent. For example, consider the path of length 2. Each of the edges is an independent matching, but the union of those edges is not even matchable (so is clearly not independent). The following lemma summarizes two ways of combining independent subgraphs that do preserve independence.

Lemma 4.2. *The following are provable in RCA_0 . (1) The union of two disjoint independent subgraphs is independent. (2) If I_1 is an independent subgraph of G and I_2 is independent in $G \setminus I_1$, then $I_1 \cup I_2$ is independent in G .*

Proof. The second property is from Aharoni, Lemma 4.4 in [6]. Assume that I_1 is an independent subgraph of G and I_2 is independent in $G \setminus I_1$.

Suppose for a contradiction that $I_1 \cup I_2$ is not independent in G . Note that $I_1 \cup I_2$ has a perfect matching, namely the union of a perfect matching of I_1 and a perfect matching of I_2 . So $I_1 \cup I_2$ must have a perfect matching M that is not independent. Let M_1 and M_2 be the matching M restricted to I_1 and I_2 , respectively. (Clearly M_1 is a perfect matching of I_1 , and M_2 is a perfect matching of I_2 .) Then there is a proper M -augmenting path P in G starting at $s \in V(G) \setminus V(M)$. If P is disjoint from I_1 , then this contradicts the independence of I_2 in $G \setminus I_1$. So consider $I_1 \cap P$, which must be nonempty. There must be a vertex $s' \in V(G) \setminus V(M_1)$ adjacent to one of the vertices in this part of the path. The restriction of P to $I_1 \cup \{s'\}$ cannot be infinite inside I_1 because that would be a proper M_1 -augmenting path, but also cannot leave I_1 for the same reason.

The first part easily follows from (2). □

For clarity, we will distill the “graph theoretic core” of the proof of PM in [5] into the following lemma.

Lemma 4.3 (Adapted from [5]). *The following hold over the given system.*

1. (RCA₀) *If G satisfies condition (A) and if M is an independent matching of G , then $G \setminus V(M)$ satisfies condition (A).*
2. (Π_2^1 -CA₀) *Suppose G satisfies condition (A) and $s \in V(G)$, but there are no nonempty independent subgraphs of G . Then there is a matching M'' of G such that $s \in V(M'')$ and $G \setminus V(M'')$ satisfies condition (A).*

To prove PM using Lemma 4.3, Steffens’ proof first obtains a maximal independent matching. By Lemma 4.3 part (1), the graph of the remaining unmatched vertices continues to satisfy condition (A). Also, by Lemma 4.2 part (2), the remaining graph has no nonempty independent subgraphs. Thus by Lemma 4.3 part (2), we can find a matching M'' of G that covers any one of the remaining unmatched vertices, and so that its removal preserves condition (A). Iterating this process, all vertices of G can be covered by the edges of a matching.

Because our statement of Lemma 4.3 is not exactly identical to any lemma of Steffens, we close our discussion with a short sketch of its proof, which simply indicates how it follows from the lemmas of Steffens [5].

Proof sketch of Lemma 4.3. For the first property, let M' be a matching of $G \setminus V(M)$, and let $s \in V(G) \setminus V(M)$ be a vertex not covered by M' . Since $V(M)$ and $V(M')$ are disjoint, $M \cup M'$ is a matching of G . Then, because G satisfies condition (A), there must be a path P in G that starts at s and augments $M \cup M'$. Now it cannot be that P ever enters M , for if it did, the restriction of P to M together with the vertex s' before P first enters M and the first vertex of P no longer in M (if there is one) would be a proper M -augmenting path. Since M is independent, this cannot happen. Thus P is disjoint from M , so it is in fact an M' -augmenting path contained in $G \setminus V(M)$, as needed.

For the second property, let M' be a maximal (possibly empty) independent matching of $G \setminus s$ (this requires an application of MIM, which, in Theorem 4.7, we will show is provable in $\Pi_2^1\text{-CA}_0$). Because G satisfies condition (A), and because $s \notin V(M')$, there is an M' -augmenting path P starting at s . Let $M'' = M' \Delta P$, which is a perfect matching of $V(M') \cup P$. Together, Lemmas 5 and 7 of [5] are exactly the statement that $G \setminus V(M'')$ satisfies condition (A). Furthermore, it is straightforward to see that the proofs of Lemmas 5 and 7 of [5] can be formalized inside ACA_0 . \square

4.1. Proofs of MIM and PM

Given a graph G , Steffens' original proof of PM used infinitely many applications of Zorn's Lemma to find a matching of G . More precisely, Steffens used Zorn's Lemma to prove the existence of maximal independent matchings (our MIM) and then recursively applied that principle ω times. In this section, we will first show that MIM is provable in $\Pi_2^1\text{-CA}_0$ using an inner model technique. Later, we will use a related line of reasoning to see that it is not possible for either PM or MM to imply $\Pi_2^1\text{-CA}_0$.

The inner model technique we use is analogous to the ones used in [3] to prove that ATR_0 implies CKDT and in [2] to prove that $\Pi_1^1\text{-CA}_0$ implies Menger's theorem for countable webs. The primary difference is that while the above proofs were able to use ω - or β -models; we use β_2 -models because we will need to reflect the existence of a *maximal* independent matching (a Σ_2^1 property) out of the model, and still have it be maximal independent. We begin with some background on β and β_2 models.

Definition 4.4 (Simpson [11]). The following definition is made within RCA_0 . A *countable coded ω -model* is a set $W \subseteq \mathbb{N}$, viewed as encoding the L_2 -model $\mathcal{M} = (\mathbb{N}, S_{\mathcal{M}}, +, \cdot, 0, 1, <)$ with $S_{\mathcal{M}} = \{(W)_n : n \in \mathbb{N}\}$.

Let $0 \leq k < \omega$. A β_k -model is an ω -model \mathcal{M} such that for all Σ_k^1 sentences φ with parameters from \mathcal{M} , φ is true if and only if $\mathcal{M} \models \varphi$. A *countable coded β_k -model* is a countable coded ω -model \mathcal{M} such that for all $e, m \in \mathbb{N}$ and $X, Y \in S_{\mathcal{M}}$, $\varphi_k(e, m, X, Y)$ is true if and only if $\mathcal{M} \models \varphi_k(e, m, X, Y)$, where $\varphi_k(e, m, X, Y)$ is a universal Σ_k^1 formula (see [11]). A *countable coded β -model* is a countable coded β_1 -model.

To understand the semantic meaning of Definition 4.4, let $\mathcal{N} = (\mathbb{N}, S_{\mathcal{N}})$ be any model of RCA_0 . Then a set $W \in S_{\mathcal{N}}$ defines a countable coded β_k -model, in the context of the intended model $(\mathbb{N}, S_{\mathcal{N}})$, if for each Σ_k^1 sentence ϕ_k with set parameters from $\{(W)_n : n \in \mathbb{N}\}$, $(\mathbb{N}, S_{\mathcal{N}}) \models \phi_k$ if and only if $(\mathbb{N}, \{(W)_n : n \in \mathbb{N}\}) \models \phi_k$.

By combining Theorem VII.7.4 and Theorem VII.6.9(3) of [11], we see that, over ACA_0 , $\Pi_1^1\text{-CA}_0$ (resp., $\Pi_2^1\text{-CA}_0$) is equivalent to the statement that for all $X \subseteq \mathbb{N}$, there exists a countable coded β -model (resp., countable coded β_2 -model) \mathcal{M} such that $X \in \mathcal{M}$.

To build a maximal independent matching, we require Lemmas 1 and 2 of [5].

Lemma 4.5 (Lemma 1 of Steffens [5]). *Let G be a countable graph. RCA_0 proves that if M is a perfect matching of G , and G' is an independent subgraph of G , then there is no edge $\{s, v\} \in M$ such that $s \in V(G) \setminus V(G')$ and $v \in V(G')$.*

Proof. Suppose for a contradiction that there is an edge $\{s, v\} \in M$ with $s \in V(G) \setminus V(G')$ and $v \in V(G')$, and (by definition of independence of G') let M' be an independent perfect matching of G' . We can now form a proper M' -augmenting path starting at s , which will contradict the independence of M' . Start with the edge $\{s, v\} \in M$, and let $v_0 = s$ and $v_1 = v$. By assumption, $v_1 \in V(M')$, so there is an edge in M' from v_1 to some $v_2 \neq v_1$. Note that in M , v_2 cannot be matched to v_1 (because v_1 is matched to $v_0 \neq v_2$) and is thus matched to another vertex v_3 . Repeating this process, we obtain a path P that either keeps going forever, or it leaves G' . In either case, it is a proper M' -augmenting path. Finally, note that the definition of P is Δ_1^0 in $M \oplus M'$, so it exists by RCA_0 . \square

Lemma 4.6 (Lemma 2 of Steffens [5]). *Let G be a countable graph. RCA_0 proves that a subgraph G' of G is independent if and only if there exists an independent perfect matching for G' .*

Proof. The forward direction follows trivially from the definition of independent subgraph. For the other direction, let G' be a subgraph of G and assume that there exists an independent perfect matching M' for G' . We must show that every perfect matching of G' is also independent. Suppose toward a contradiction that there is a perfect matching M of G' that is not independent. Then there is a proper M -augmenting path P beginning at some $s \in V(G) \setminus V(G')$, which has second vertex $v \in V(G')$.

Let $v_0 = s$ and $v_1 = v$. Similar to the proof of Lemma 4.5, define a path $(v_i)_{i < k \leq \omega}$ such that $\{v_1, v_2\} \in M'$, $\{v_2, v_3\} \in M$, and so forth, alternating between the two matchings. This is now a proper M' -augmenting path starting at s , which contradicts the independence of M' . \square

We can now prove the first main result of this section. Note that if $P(G)$ is a property of a graph G , then we sometimes write $P(G)^{\mathcal{M}}$ as an abbreviation for $\mathcal{M} \models P(G)$.

Theorem 4.7. $\Pi_2^1\text{-CA}_0$ *proves* MIM.

Proof. By $\Pi_2^1\text{-CA}_0$, there is a countable coded β_2 -model \mathcal{M} containing the graph G as an element. Note that it is arithmetical with parameter \mathcal{M} to define independent $^{\mathcal{M}}$ matchings, as these are the columns of \mathcal{M} which are matchings that are not properly augmented by any other column of \mathcal{M} . By arithmetical comprehension, using a code for \mathcal{M} as a parameter, we can form the set of all independent $^{\mathcal{M}}$ matchings in \mathcal{M} . Arithmetically relative to this oracle, form a sequence $\{M_i\}_{i \in \mathbb{N}}$ of independent $^{\mathcal{M}}$ matchings in \mathcal{M} , with increasing support, by recursion on n as follows. Let g_n denote the n -th vertex in G . M_0 is the first independent $^{\mathcal{M}}$ matching that contains g_0 if it exists, otherwise it is empty. If there is an independent $^{\mathcal{M}}$ matching that extends the support of M_n and matches g_{n+1} , set it to be M_{n+1} . Otherwise, set $M_{n+1} = M_n$. Finally, we form the chain $\{M_i^*\}_{i \in \mathbb{N}}$ by setting $M_0^* = M_0$ and $M_{i+1}^* = (M_{i+1} \setminus E(V(M_i))) \cup M_i^*$. Then define $M^* = \bigcup_{i \in \mathbb{N}} M_i^*$, as in Steffens' proof.

We claim that each M_i^* is independent $^{\mathcal{M}}$. Clearly M_0^* is independent $^{\mathcal{M}}$. So assume inductively that M_i^* is independent $^{\mathcal{M}}$, and we will show that M_{i+1}^* is independent $^{\mathcal{M}}$.

Set $I_1 = V(M_i^*)$ and $I_2 = V(M_{i+1}^* \setminus M_i^*)$. Then M_i^* is a perfect matching of I_1 and is independent $^{\mathcal{M}}$ in G by hypothesis. Also, $M_{i+1}^* \setminus M_i^*$ is a perfect matching of I_2 , and we claim it is independent $^{\mathcal{M}}$ in $G \setminus I_1$. Indeed, suppose toward a contradiction that there is a proper $(M_{i+1}^* \setminus M_i^*)$ -augmenting path

P in $G \setminus I_1$ starting outside of I_2 . By our choice of M_{i+1}^* , P is also a proper M_{i+1} -augmenting path starting outside of $V(M_{i+1})$. This is a contradiction, because M_{i+1} is independent $^{\mathcal{M}}$ in G . Therefore, by Lemma 4.6, I_1 and I_2 are independent $^{\mathcal{M}}$ subgraphs of G and $G \setminus I_1$, respectively. So $I_1 \cup I_2$ is an independent $^{\mathcal{M}}$ subgraph of G by Lemma 4.2(2). Since M_{i+1}^* is a perfect matching of $I_1 \cup I_2$, it is an independent $^{\mathcal{M}}$ matching of G , as desired.

We claim that the union M^* is truly independent (outside of \mathcal{M}). If it were not, then there would be a proper M^* -augmenting path P starting at a vertex $v \in V(G) \setminus V(M^*)$. Because the M_i^* form a chain of matchings of increasing support, if P properly augments M^* , then for some i , the restriction P_i of P to M_i^* (the path P_i is in \mathcal{M} because it is definable from M_i^*) will properly augment M_i^* . Because M_i^* exists in \mathcal{M} , because M_i^* is augmented by P_i (a Σ_1^1 property with parameters in \mathcal{M}), and because \mathcal{M} is a β_2 -model (so satisfies Σ_2^1 reflection), some proper M_i^* -augmenting path exists in the model. But this contradicts the fact that each M_i^* was independent $^{\mathcal{M}}$.

We also claim that M^* is maximal independent. Suppose it were not, for a contradiction. Then there would be an independent matching whose vertex set properly contains $V(M^*)$. Let g_n be the vertex of least index not in $V(M^*)$ that is in the larger independent matching. Now we have

$$\exists Y [Y \text{ is independent} \wedge V(Y) \supsetneq V(M_n^*) \wedge g_n \in V(Y)].$$

Because the statement “ Y is independent” is Π_1^1 , the above is a Σ_2^1 sentence.

Since \mathcal{M} is a β_2 -model and the above Σ_2^1 sentence is true outside of \mathcal{M} , it must be true in \mathcal{M} (and this is the point where we use the full power of the β_2 -model). However, recall that by our definition of the M_i , there must not be an independent $^{\mathcal{M}}$ matching which extends M_n and also contains g_n (and $V(M_n) = V(M_n^*)$). So we have a contradiction, and therefore M^* is a maximal independent matching of G . \square

One might wonder if the proof above could be simplified by using a less complex (than Π_1^1) way of saying that a matching is independent. However, using an argument similar to those in Section 5, it is easy to see that deciding whether a given matching in a computable graph is independent is Π_1^1 -complete.

It is now possible to give an upper bound on the strength of PM. The system $\Pi_2^1\text{-CA}_0^+$ is the system that permits ω -many iterated applications of $\Pi_2^1\text{-CA}_0$.

Theorem 4.8. $\Pi_2^1\text{-CA}_0^+$ proves PM.

Proof. The idea of the proof is the same as Steffens'. Let G be a graph that satisfies condition (A) and suppose $V(G) = \{v_0, v_1, \dots\}$. Apply MIM to obtain a maximal independent matching M_0 of G . Define $G_1 = G \setminus V(M_0)$, which continues to satisfy condition (A) by Lemma 4.3. Apply the second part of Lemma 4.3 (which entails another application of MIM) to obtain a matching M_1 contained in G_1 which matches the vertex of least index not matched by M_0 , such that $G_2 = G_1 \setminus V(M_1)$ still satisfies condition (A). At this point, we do not know whether G_2 has nontrivial independent subgraphs (or unmatched vertices), so we start the process over again with G_2 as the new G .

Iterate the above process to obtain a sequence of matchings $\{M_n\}_{n \in \mathbb{N}}$, with pairwise disjoint vertex sets, such that every vertex of G is matched by some M_i . We see that $M = \bigcup_{n \in \mathbb{N}} M_n$ is the desired perfect matching of G . Notice that we have potentially applied MIM ω -many times in the above construction. \square

This proof appears to do more work than is actually needed. Because PM is *true*, the graph G has a perfect matching M . Because M covers each vertex of G , it trivially satisfies the definition of an independent matching, so M is an independent matching of maximal support. But that means that, as long as PM is true, the first application of MIM in its proof already yields the perfect matching of G . The rest of the construction (which involved finding infinitely many new maximal independent matchings) was, *in retrospect*, unnecessary. We will return to this observation in Section 4.2.

Indeed, the complexity of PM means that this upper bound cannot be sharp. We will use a generalization of a well known fact, following the presentation of Marcone [13]. Although we will only need the cases for $k = 1$ and $k = 2$ in this paper, we include a general statement of the property for completeness.

Proposition 4.9. *Let $k \geq 1$. No Π_{k+1}^1 statement that is consistent with ATR_0 can imply $\Pi_k^1\text{-CA}_0$, even over ATR_0 .*

Proof. Consider the sentence $\forall X \psi(X)$, where $\psi(X)$ is Σ_k^1 and suppose that the theory T consisting of ATR_0 and $\forall X \psi(X)$ is consistent.

Suppose toward a contradiction that $\forall X \psi(X)$ implies $\Pi_k^1\text{-CA}_0$ over ATR_0 . Then T proves the existence of a countable-coded β_k -model \mathcal{M} . By definition,

$\mathcal{M} \models \psi(X)$ for each $X \in \mathcal{M}$, so $\mathcal{M} \models \forall X \psi(X)$. Since every β -model (and so every β_k -model) is a model of ATR_0 , we have that \mathcal{M} is a model of T .

But by Gödel's Second Incompleteness Theorem, no consistent theory T which implies $\Pi_1^1\text{-CA}_0$ can prove its own consistency, contradicting the assumption that $\forall X \psi(X)$ implies $\Pi_k^1\text{-CA}_0$ over ATR_0 . \square

Recall that the statement of PM itself has the form

$$\forall X [(\forall Y_1)(\exists Y_2)\theta(X, Y_1, Y_2) \rightarrow \exists Z \psi(X, Z)],$$

with θ, ψ arithmetical, and so is equivalent (after prenexing) to a Π_3^1 formula. Also, since $\Pi_2^1\text{-CA}_0^+$ implies PM, it is consistent with ATR_0 . Thus we get the following immediately from Proposition 4.9.

Corollary 4.10. *PM does not imply $\Pi_2^1\text{-CA}_0$, even over ATR_0 .*

4.2. A proof of MM

We now turn our attention to the strongest version of PM studied in this paper, the statement that *every* graph has a matching of maximal support. To prove this, we will need another lemma of Steffens.

Lemma 4.11 (Lemma 5 of Steffens [5]). *The following is provable in ACA_0 . A graph G satisfies condition (A) iff for every independent subgraph G' of G and for every vertex $s \in V(G) \setminus V(G')$, there exists a vertex $v \in V(G) \setminus V(G')$ such that $\{s, v\} \in E(G)$.*

Proof. It is straightforward to check that the proofs of Steffens' Lemmas 4 and 5 from [5], the latter relying on the former, both hold in ACA_0 . \square

Theorem 4.12. *MIM + PM implies MM over RCA_0 .*

Proof. Fix a countable graph G . Applying MIM, we obtain a maximal independent subgraph $I \subseteq G$ and a corresponding independent matching M .

If $I = G$, then we have a perfect (hence maximal) matching and we are done. So suppose $I \neq G$, and let $N \subseteq V(G) \setminus V(I)$ be the set of vertices with no neighbors in $G \setminus I$. (The vertices in N are not necessarily isolated in G as they may have neighbors in I itself.) Consider the subgraph $H = V(G) \setminus (V(I) \cup N)$. We claim that H does not have any nonempty independent subgraphs. Indeed, since any nonempty independent subgraph G' of H is also a nonempty independent subgraph of $G \setminus I$, G' could therefore be combined with I to form a new, larger independent subgraph, by

Lemma 4.2(2), which would contradict the maximality of I . Therefore the only independent subgraph of H is the empty subgraph, and clearly each vertex in H has a neighbor in H , so H satisfies condition (A) by Lemma 4.11.

Applying PM, we obtain a perfect matching \widehat{M} of H . First, note that $M \cup \widehat{M}$ is a matching because $V(M)$ and $V(\widehat{M})$ are disjoint sets of vertices. We wish to show that $M \cup \widehat{M}$ is a matching of G of maximal support. Suppose that $\{s, v\}$ is an edge covering one of the vertices of N . By definition of N , the second vertex must be contained in $I = V(M)$, where M was an independent matching. But this contradicts Lemma 4.5, applied to the graphs G and I . Thus, $M \cup \widehat{M}$ must be a maximal matching of G . \square

Theorem 4.12 is important for two reasons. First, it allows us to use the upper bounds for PM and MIM to obtain an upper bound for MM. Second, and more surprisingly, the fact that MM has lower sentence complexity than MIM will allow us to separate both principles from $\Pi_2^1\text{-CA}_0$.

Corollary 4.13. $\Pi_2^1\text{-CA}_0^+$ proves MM over RCA_0 , and MM does not imply $\Pi_2^1\text{-CA}_0$, even over ATR_0 .

Proof. $\Pi_2^1\text{-CA}_0^+$ proves PM by Theorem 4.8, and $\Pi_2^1\text{-CA}_0$ proves MIM by Theorem 4.7. Thus, $\Pi_2^1\text{-CA}_0^+$ proves MM by Theorem 4.12.

Note that MM is the following Π_3^1 sentence: for each G , there is an M such that for any other M' , if M and M' are both matchings, then $V(M)$ is not strictly contained in $V(M')$. Therefore, since MM is true and thus consistent with ATR_0 , MM does not imply $\Pi_2^1\text{-CA}_0$ by Proposition 4.9. \square

Because maximal matchings are independent, the upper bound on MM also applies to MIM.

Proposition 4.14. MM implies MIM over RCA_0 .

Proof. Any maximal matching M of a graph G must be independent. To see why, consider any M -augmenting path P . If P were a proper augmenting path, then $M \Delta P$ would be a matching with strictly greater support than M , which contradicts the fact that M is a maximal matching. Because every independent matching is also a matching, the existence of matchings of maximal support implies the existence of maximal independent matchings. \square

Corollary 4.15. MIM does not imply $\Pi_2^1\text{-CA}_0$, even over ATR_0 .

Proof. By Proposition 4.14, MM implies MIM over RCA_0 . Since MM does not imply $\Pi_2^1\text{-CA}_0$, it follows that MIM cannot imply $\Pi_2^1\text{-CA}_0$. \square

It is not clear whether any of MM, MIM, or PM are equivalent over RCA_0 . To shed light on this, note that the proof of MM required *both* PM *and* MIM. However, only a very special case of PM is used in the proof. It is natural to ask exactly how much of PM is actually used in the proof of MM.

Statement 4.16. Let (\star) denote the statement “if G satisfies condition (A) and G has no nonempty independent subgraphs, then G is empty.”

Note that (\star) is exactly the statement that the first maximal independent matching obtained in the proof of PM is a perfect matching of the graph, since it asserts that $G \setminus V(M) = \emptyset$. This tells us immediately that $\text{MIM} + (\star)$ implies PM. By Theorem 4.12, we immediately get the following.

Corollary 4.17. $\text{MIM} + (\star)$ implies MM over RCA_0 .

Actually, (\star) can simply replace PM in the proof of Theorem 4.12, since we are really only using (\star) when we prove that the maximal independent matching is itself maximal. To see why, recall that the subgraph H from the proof of Theorem 4.12 has no nonempty independent subgraphs and has no isolated vertices, so satisfies condition (A) by Lemma 4.11. By (\star) , H is the empty subgraph so its perfect matching \widehat{M} exists (and is empty), as needed in the proof of Theorem 4.12.

In fact, (\star) really is a special case of PM.

Proposition 4.18. PM implies (\star) over RCA_0 .

Proof. Let G be a graph which satisfies condition (A) and has no nonempty independent subgraphs. By PM, G has a perfect matching M . In other words, $V(M) = V(G)$. In addition, note that M is maximal, so M is an independent matching by the proof of Proposition 4.14, and so $V(M)$ is an independent subgraph of G . By assumption, G has no nonempty independent matchings, and so we must have $V(M) = \emptyset$. Because $V(M) = V(G)$, G must be empty. \square

On the face of it, (\star) is a straightforward, true principle. Thus, it is reasonable to conjecture that it has a proof in $\Pi_1^1\text{-CA}_0$. We will show in Theorem 5.6 that MIM implies $\Pi_1^1\text{-CA}_0$, so such a proof of (\star) would imply

the equivalence of MIM and MM. In addition, if (\star) is provable in $\Pi_1^1\text{-CA}_0$, then it may be possible to prove PM by iterating a contrapositive version of (\star) transfinitely many times.

Conjecture 4.19. *MM and MIM are equivalent over RCA_0 , $\Pi_1^1\text{-TR}_0$ proves PM, and $\Pi_2^1\text{-CA}_0$ proves MM.*

Although apparently simple, (\star) is surprisingly difficult to either prove or to code into. No direct proof of (\star) is known to the authors, other than its proof from PM.

Question 4.20. What is the exact strength of (\star) ? Does (\star) imply PM, or can (\star) be separated from PM using a forcing construction?

Recall that $\text{MIM} + (\star)$ implies PM. Thus even a proof of (\star) in $\Pi_2^1\text{-CA}_0$ would yield an improved upper bound on PM and hence on MM.

5. Lower bounds for matchings in general

In this section, we establish lower bounds on the complexity of the principles whose upper bounds were given above.

In many of these reversals, we uniformly convert a tree T into what we will call its *doubling tree* \widehat{T} . The idea is that each vertex v in T *except* the root gets replaced by two vertices, connected by an edge (we call this a *doubling edge*). The bottom vertex of that edge is a child of the parent of v ; the top vertex of the edge is the parent of every child of v . Note that Aharoni, Magidor, and Shore [1] also construct this type of tree (in their Theorem 4.13). Clearly T has an infinite path if and only if \widehat{T} has an infinite path. More importantly, there is a correspondence between paths in T , matchings of \widehat{T} , and condition (A).

Lemma 5.1. *Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree, and let \widehat{T} be its doubling tree. Then the following are equivalent over RCA_0 :*

1. T has an infinite path.
2. \widehat{T} has a perfect matching.
3. \widehat{T} satisfies condition (A).

Proof. (1 \rightarrow 2). Suppose there is an infinite path P through T . We define a perfect matching M of \widehat{T} as follows. For every edge in P , put the corresponding edge from \widehat{T} into M . M is a matching since the corresponding path

in \widehat{T} alternates between edges in T and the added doubling edges. For each vertex not on P , include its doubling edge in M (the root is on P , so every vertex not on P is adjacent to exactly one doubling edge). Note that no two doubling edges are adjacent in \widehat{T} . Now every vertex in \widehat{T} is matched by M , making it a perfect matching.

(2 \rightarrow 3). This holds in RCA_0 for all graphs (Proposition 1.3).

(3 \rightarrow 1). Let M be the matching consisting of exactly the doubling edges in \widehat{T} . This leaves precisely the root r of \widehat{T} unmatched. By condition (A), there is an M -augmenting path starting at r . Since no other vertex is unmatched, this path must be infinite, and corresponds to an infinite path back in T . \square

5.1. Lower bounds for PM

Theorem 5.2. *PM implies $\Sigma_1^1\text{-ACA}_0$ over RCA_0 .*

Proof. In Proposition 3.5, we proved that the restriction of PM to locally finite graphs is equivalent to ACA_0 , so we may work over ACA_0 . Let $\langle T_k : k \in \mathbb{N} \rangle$ be a sequence of trees $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $\forall k [T_k \text{ has an infinite path}]$. Let $\langle \widehat{T}_k : k \in \mathbb{N} \rangle$ be the associated sequence of doubling trees, and define $G = \bigsqcup_{k \in \mathbb{N}} \widehat{T}_k$. By Theorem V.1.7' of [11], it suffices to show there exists a sequence $\langle g_k : k \in \mathbb{N} \rangle$ so that

$$\forall k [g_k \text{ is an infinite path through } T_k].$$

We claim that G satisfies condition (A). In the proof of Lemma 5.1, we saw that \widehat{T}_i satisfies condition (A) if and only if T_i has an infinite path. Since each T_i has an infinite path, \widehat{T}_i satisfies condition (A) for each $i \in \mathbb{N}$. It is easy to see, working in ACA_0 , that a disjoint union of graphs satisfying condition (A) also satisfies condition (A), and therefore G must satisfy condition (A). This proves the claim.

Then by PM, there is a perfect matching M of G . This allows us to uniformly define an infinite path P_k through the doubling tree \widehat{T}_k as follows. Starting at the root, follow the matching up the tree. The root vertex is matched by M to exactly one child. That vertex has only one child, via its doubling edge. This child is matched by M to exactly one of its children, and so on. By restricting each P_k to T_k , we obtain a sequence $\langle g_k : k \in \mathbb{N} \rangle$, where g_k is an infinite path through T_k for each $k \in \mathbb{N}$. \square

In the proof above, we do not appear to use the full strength of PM. As we have already noted, some of the complexity of PM appears to arise from the

complexity of deciding if condition (A) holds. But in the proof above, it was computable to show that each \widehat{T}_i satisfied condition (A), and arithmetical to show that the union of a sequence of pairwise disjoint graphs, all of which satisfy condition (A), satisfies condition (A).

It is important to note that deciding whether condition (A) holds for countable graphs is much harder in general. Indeed, deciding whether a computable graph satisfies condition (A) is Σ_1^1 -hard.

Proposition 5.3. *The set of indices of computable graphs which satisfy condition (A) is Σ_1^1 -hard.*

Proof. Since the set of indices for trees with infinite paths is Σ_1^1 -complete (by Theorem 16.XX in [9]), it suffices to show that given a computable tree T , we can find a computable graph G , such that T has an infinite path if and only if G satisfies condition (A). Given T , let $G = \widehat{T}$, the doubling tree of T . By Lemma 5.1 we have that \widehat{T} satisfies condition (A) if and only if T has an infinite path, and we are done. \square

While condition (A) has a Π_2^1 definition, it cannot be Π_2^1 -complete. To see why, recall the statement of PM: G satisfies condition (A) if and only if G has a perfect matching. Since deciding whether G has a perfect matching is Σ_1^1 , it is thus possible to decide if condition (A) holds using a Σ_1^1 statement.

For at least some classes of graphs, condition (A) is equivalent over RCA_0 to a sentence of simpler complexity. In the special case of doubling tree graphs, Lemma 5.1 says that RCA_0 proves the equivalence between satisfying condition (A) and having a perfect matching. Since the disjoint union of graphs satisfying condition (A) continues to satisfy condition (A), the restriction of PM to disjoint unions of doubling trees is actually Π_2^1 .

More generally, the reversal from PM to $\Sigma_1^1\text{-AC}_0$ can be seen as using only the statement that the union of a sequence of pairwise disjoint graphs, all of which have perfect matchings, has a perfect matching. This can be viewed as a Π_2^1 version of PM, which we refer to as Collection of PM.

Corollary 5.4. *The following are equivalent over RCA_0 :*

1. $\Sigma_1^1\text{-AC}_0$
2. (Collection of PM) *Fix a sequence of pairwise disjoint graphs $\langle G_i \rangle$. If G_i has a perfect matching for each i , then $G = \bigsqcup G_i$ has a perfect matching.*

Proof. (1 \rightarrow 2) is an easy application of $\Sigma_1^1\text{-AC}_0$. (2 \rightarrow 1) is a consequence of the proofs of Proposition 3.5 (which shows that Collection of PM implies ACA_0) and Theorem 5.2. □

Of course, Steffens selected condition (A) for its graph-theoretic content, rather than for its formula complexity.

Question 5.5. Is there a natural Δ_2^1 statement equivalent to condition (A), which does not involve simply checking for the existence of a perfect matching?

If this is the case, it would yield a version of full PM whose complexity would be Π_2^1 , and hence this version of PM would not imply $\Pi_1^1\text{-CA}_0$, even over ATR_0 (Proposition 4.17 from [1]).

5.2. The strength of MIM and Sequential PM

Theorem 5.6. *MIM implies $\Pi_1^1\text{-CA}_0$ over RCA_0 .*

Proof. First, we claim that MIM implies ACA_0 . Using the same construction as in the proof of Proposition 3.5, we get a graph consisting of disjoint paths of length 1 or 3. Since each disjoint path has as its maximal independent matching a perfect matching, the maximal independent matching of the entire graph will be its perfect matching, from which we can define the range of the given function.

We may now work over ACA_0 . Let $\langle T_i : i \in \mathbb{N} \rangle$ be a sequence of trees $T_i \subseteq \mathbb{N}^{<\mathbb{N}}$. We wish to define a set Z such that $i \in Z$ if and only if T_i has an infinite path. Form the associated sequence $\langle \widehat{T}_i : i \in \mathbb{N} \rangle$ of doubling trees. Let G be the disjoint union $G = \bigsqcup \widehat{T}_i$, and by MIM, let M be a maximal independent matching for G .

Consider M restricted to \widehat{T}_i . Note that if this is not a perfect matching, then \widehat{T}_i cannot have any perfect matching, since such a perfect matching would be an independent matching of larger support. Thus M restricted to \widehat{T}_i is a perfect matching if and only if \widehat{T}_i has a perfect matching, which by Lemma 5.1 occurs if and only if T_i has an infinite path.

Since ACA_0 can form the set of i such that M restricted to \widehat{T}_i is a perfect matching of \widehat{T}_i , we get the desired set Z . □

Recall from Statement 2.1, Sequential PM asserts that for each sequence of disjoint graphs $\langle G_i : i \in \mathbb{N} \rangle$, there is a sequence of matchings $\langle M_i : i \in \mathbb{N} \rangle$ such that for all i , if G_i satisfies condition (A), then M_i is a perfect matching of G_i .

Sequential PM clearly implies PM over RCA_0 , and it is natural to conjecture that Sequential PM is strictly stronger than PM. Although less common in reverse mathematics, principles like Sequential PM sometimes occur in the study of Weihrauch principles. There, Sequential PM is called the “parallelization” of the “total continuation” of PM (see [14], Section 8, for another similar principle).

Proposition 5.7. *MM implies Sequential PM over RCA_0 .*

Proof. Assume MM, and let $S = \langle G_i : i \in \mathbb{N} \rangle$ be a sequence of pairwise disjoint graphs. Let $G' = \bigsqcup G_i$ be the effective disjoint union of the columns of S , and apply MM to obtain a maximal matching M of G' . For each i , let M_i be the restriction of M to G_i . To verify that $\langle M_i : i \in \mathbb{N} \rangle$ is the desired sequence of edge sets, fix i and suppose $G_i = (V, E)$ satisfies condition (A). For a contradiction, assume M_i is not a perfect matching of G_i . Then there is a vertex $s \in V \setminus V(M_i)$. Since G_i satisfies condition (A), there is an M_i -augmenting path P starting at s . So $M_i \Delta P$ is a matching of G_i that improves the support of M_i , contradicting the fact that M is maximal. \square

In fact, our reversal from MIM to $\Pi_1^1\text{-CA}_0$ also goes through in Sequential PM.

Proposition 5.8. *Sequential PM implies $\Pi_1^1\text{-CA}_0$ over RCA_0 .*

Proof. As Sequential PM implies PM which implies ACA_0 (all over RCA_0), we may work over ACA_0 . Let $\langle T_i : i \in \mathbb{N} \rangle$ be a sequence of trees. It suffices to show that the set $I = \{i : T_i \text{ has an infinite path}\}$ exists.

For each tree T_i , consider its doubling tree \widehat{T}_i . Applying Sequential PM to $\langle \widehat{T}_i : i \in \mathbb{N} \rangle$, we obtain a sequence $\langle M_i : i \in \mathbb{N} \rangle$ of edge sets with the property that if \widehat{T}_i satisfies condition (A), then M_i is a perfect matching of \widehat{T}_i .

Since it is arithmetical to decide if a given M_i is a perfect matching, we can form $\{i : M_i \text{ is a perfect matching of } \widehat{T}_i\}$ in ACA_0 . By Lemma 5.1, this is the set of i such that T_i has an infinite path. \square

In the previous section, we showed that MIM is provable in $\Pi_2^1\text{-CA}_0$, that MM is provable in $\Pi_2^1\text{-CA}_0^+$, and conjectured that MM is provable in $\Pi_2^1\text{-CA}_0$.

Question 5.9. What is the exact strength of MM and, separately, of MIM?

6. Matchings for graphs with no infinite paths

In this final section, we will refine our analysis of PM by considering a special case, whose strength is in the region of ATR_0 . We write Finite Path PM to denote the restriction of PM to graphs with only finite paths.

Recall that much of the complexity of PM comes from the complexity of condition (A). In the case of Finite Path PM, condition (A) is equivalent to a Π_1^1 formula (with the graph as a parameter), and Finite Path PM is equivalent to a Π_2^1 sentence. Moreover, the notion of “independent matching” becomes arithmetical. These reductions in complexity result in significantly improved upper bounds.

Theorem 6.1. $\Pi_1^1\text{-CA}_0$ proves MIM for graphs without infinite paths.

Proof. Let \mathcal{M} be a countable coded β -model containing G . Using ACA_0 and a code for \mathcal{M} as a parameter, construct a maximal independent matching as described in our proof of full MIM (Theorem 4.8) which used a β_2 -model. The key difference is that now the property of a matching being independent is arithmetical in the matching and the graph, and therefore the formula witnessing a counterexample to a matching being maximal independent is Σ_1^1 (with parameters from the model), and therefore reflects into or out of a β -model. The rest of the proof is analogous. \square

Corollary 6.2. $\Pi_1^1\text{-CA}_0^+$ proves Finite Path PM.

Proof. Because there are no infinite paths in G , the property of a matching being independent is arithmetical and the property of a graph satisfying condition (A) is Π_1^1 . Therefore the proof is analogous to the proof given earlier of PM in Theorem 4.8, this time using countable coded β -models. \square

Because Finite Path PM is equivalent to a Π_2^1 sentence, Proposition 4.9 with $k = 1$ shows that this upper bound is not optimal.

Corollary 6.3. Finite Path PM does not imply $\Pi_1^1\text{-CA}_0$, even over ATR_0 .

6.1. Perfect matchings code hyperarithmetical sets

To obtain lower bounds, we will code hyperarithmetical sets into Finite Path PM. We assume a basic knowledge of hyperarithmetical theory, and refer the reader to Ash and Knight [15] for additional background.

We will identify each computable ordinal with its corresponding notation in Kleene's \mathcal{O} . For each $X \subseteq \mathbb{N}$ and $e \in \mathcal{O}^X$, the set H_e^X (referred to as an H^X -set) corresponds to the hierarchy resulting from recursively taking jumps of X along the X -computable well order with notation e . As pointed out by Aharoni, Magidor and Shore in [1], ATR_0 is equivalent to

$$\forall X \forall e [e \in \mathcal{O}^X \rightarrow H_e^X \text{ exists}].$$

We will show that for each set X and for each $e \in \mathcal{O}^X$, there is an X -computable graph G_e^X which satisfies condition (A), such that any perfect matching of G_e^X computes the set H_e^X . As a result of the uniformity of our construction, it follows that ATR is Weihrauch reducible to PM (see Section 6.2). The Π_2^1 nature of condition (A) complicates the reverse-mathematical picture, which will be discussed at the end of the section.

We follow the general approach of [1], by defining graphs whose perfect matchings code the truth or falsity of *propositions*. We also recursively show how, given sentences whose truth is coded by the perfect matchings of coding graphs, to define a new graph that codes the negation, conjunction, or quantification of these sentences. We will then be able to code the relation $n \in X$ as a simple X -computable proposition. By coding negations, conjunctions, and existentials of already coded propositions, we will code the membership relation of the H^X -sets corresponding to successor ordinals. Finally, by coding sequences of already coded propositions, we will code the membership relation of the H^X -sets corresponding to limit ordinals. Appealing to effective transfinite recursion, a consequence of the recursion theorem, we obtain a single uniform computation that maps sets X and codes e to graphs G_e^X .

Definition 6.4. A *coding graph*, illustrated in Figure 2, is a tuple $\langle G, l, r, c \rangle$ such that G is a connected graph and $l, r, c \in V(G)$. We write $G^\circ = G \setminus \{l, r, c\}$ to refer to the “interior” of the coding graph. We also require that: r is adjacent to a unique vertex in $V(G^\circ)$, l is adjacent to a unique vertex in $V(G^\circ)$, and c is adjacent to l and r , but is not adjacent to any vertices in $V(G^\circ)$.

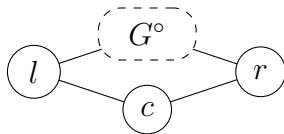


Figure 2: The coding graph $\langle G, l, r, c \rangle$

We will prove that each of the coding graphs that we recursively define has a perfect matching, and hence satisfies condition (A). In fact, we will prove that each graph we build will have a *unique* perfect matching.

Definition 6.5. Suppose that a coding graph $\langle G, l, r, c \rangle$ has a unique perfect matching. We say that this graph *codes true* if the unique matching contains the edge between G° and l , and that it *codes false* if the unique matching contains the edge between G° and r .

The central vertex c is included to ensure that the coding graph will have a matching. In each step of the recursive construction of coding graphs, we will modify the coding graphs from the previous stage, both by removing their central vertices and by adding new vertices and edges.

Lemma 6.6.

- (1) *There is a coding graph with a unique matching that codes true.*
- (2) *There is a single computable procedure which, given any coding graph with a unique matching that codes a predicate P , returns another coding graph with a unique matching that codes the negation of P .*

Proof. (1) To code “true,” we define a coding graph $\langle G, l, r, c \rangle$ with three additional vertices $\{x, y, z\}$, and edges consisting of the cycle $x-y-r-c-l-x$, together with the single edge $\{y, z\}$. This graph is illustrated in Figure 3, left. Note that the interior of this graph has vertex set $V(G^\circ) = \{x, y, z\}$. Starting with the edge $\{y, z\}$, it is easy to check that this graph has a unique perfect matching.

(2) Suppose we are given a coding graph $\langle G, l, r, c \rangle$ coding P . We define a coding graph $\langle G', l', r', c \rangle$ that codes $\neg P$ by setting $G'^\circ = G \setminus \{c\}$, connecting l to l' , and connecting r to r' . This graph is illustrated in Figure 3, right.

By assumption, $\langle G, l, r, c \rangle$ has a unique perfect matching. To see why the new graph $\langle G', l', r', c \rangle$ has a unique perfect matching, note that in any perfect matching, exactly one of l or r matches into G° . Without loss of

generality, assume that l matches into G° . Then r needs to be matched to r' , and c needs to be matched to l' , thus yielding a unique perfect matching.

Finally, we show that this new graph correctly codes $\neg P$. If the original graph G codes true, then the unique matching of G' must match l into G° , which forces $\{r, r'\}$ to be in the matching, so G' codes false as desired. On the other hand, if G codes false, then a similar argument shows G' codes true. \square

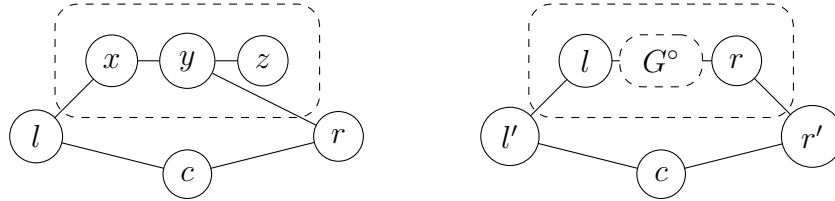


Figure 3: *Left* a graph that codes “true”. *Right*, a graph that codes negation.

Lemma 6.7. *There is a single computable procedure which, given any two coding graphs $\langle G_i, l_i, r_i, c_i \rangle$, $i \in \{1, 2\}$, each with a unique perfect matching, and where G_i codes P_i , returns another coding graph $\langle G, l, r, c \rangle$ with a unique matching, that codes $\neg P_1 \wedge P_2$.*

Together with the ability to code negations, it follows that we can construct coding graphs to code $P_1 \wedge P_2$, $P_1 \vee P_2$, and $P_1 \rightarrow P_2$.

Proof. Given the coding graphs $\langle G_i, l_i, r_i, c_i \rangle$ coding P_i , $i \in \{1, 2\}$, define a coding graph $\langle G, l, r, c \rangle$ that codes $\neg P_1 \wedge P_2$ by removing the vertices c_i , then adding a new vertex rr , and including the cycle $l_1-l_2-r_1-r_2-rr-r-c-l_1$ (Figure 4).

By assumption, the component graphs G_1 and G_2 have unique perfect matchings. It is straightforward to show that these extend to a unique perfect matching of G . There are four cases, depending on the truth value coded by the G_i .

For example, if G_1 and G_2 both code true, then l_1 matches into G_1 and l_2 matches into G_2 . Then the matching must include $\{l, c\}$, $\{r, rr\}$, and $\{r_2, r_1\}$ leading to a unique perfect matching where G codes false, matching the desired truth value of $\neg T \wedge T \equiv F$. The proofs of the other three cases are similar. \square

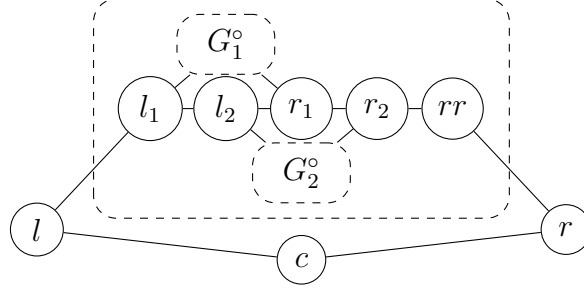


Figure 4: Coding $\neg P_1 \wedge P_2$, where P_i is coded by $\langle G_i, l_i, r_i, c_i \rangle$

In previous sections of this paper, we have often taken the union of a fixed sequence of graphs containing disjoint vertex sets. In this section, we are defining a large class of graphs by transfinite recursion. There is no guarantee that the different graphs produced in this way will have disjoint vertex sets. We will therefore use the computable join operation $\bigoplus G_i = \{\langle j, i \rangle : j \in G_i\}$ when coding sequences of propositions.

Lemma 6.8. *Let $\langle G_i \rangle$ be a sequence of coding graphs, each with a unique matching. Then $\bigoplus G_i$ can be seen as coding membership in the set $\{i : G_i \text{ codes true}\}$.*

Proof. Because the graphs induced by the computable join of the vertex sets will be disjoint, if all of the given graphs satisfy condition (A) and have perfect matchings, then the union will satisfy condition (A) and have a perfect matching. Furthermore, any perfect matching of the whole graph will be unique, and code the truth of all the component propositions. \square

Lemma 6.9. *There is a single computable procedure which, given a uniformly computable sequence of coding graphs $\langle G_n, l_n, r_n, c_n \rangle$, each with a unique matching and each coding a predicate $P(n)$, returns a coding graph with a unique matching that codes the truth of the sentence $(\exists n)P(n)$.*

Proof. For technical reasons, we must begin by using negation and conjunction to obtain a new sequence of coding graphs \widehat{G}_i which code the predicates $\widehat{P}(i) = P(i) \wedge \neg P(i-1) \wedge \dots \wedge \neg P(1)$. Then at most one of the \widehat{G}_i codes true (and if there is exactly one, it will be the \widehat{G}_i with i least possible). Note also that $(\exists n)P(n) \equiv (\exists n)\widehat{P}(n)$.

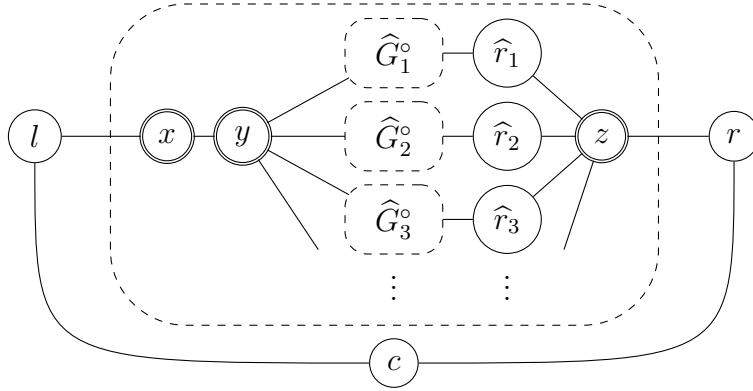


Figure 5: Coding $(\exists i)P(i)$ in the case where $P(i)$ is true for at most one i .

We can now create a new coding graph $\langle G, l, r, c \rangle$ that will code $(\exists n)\widehat{P}(n)$. We modify and combine the coding graphs \widehat{G}_i as in Figure 5: For each i , the vertices \widehat{l}_i and \widehat{c}_i are removed. The vertices of the graph being constructed consist of the remaining vertices of the coding graphs \widehat{G}_i° together with new vertices: x, y, z . For each i , the unique vertex in \widehat{G}_i° that was adjacent to \widehat{l}_i is set adjacent to the single vertex y , and all the vertices \widehat{r}_i are set adjacent to the separate vertex z . Finally, we include the path $y-x-l-c-r-z$.

We must show that there exists a unique matching of this graph, and that it codes $(\exists n)\widehat{P}(n)$. Since at most one of the predicates $\widehat{P}(i)$ is true, we have two cases.

First, suppose there exists exactly one k such that $\widehat{P}(k)$ is true. Then from the unique perfect matching of \widehat{G}_k , we get that y must be matched into \widehat{G}_k° , which means \widehat{r}_k must be matched to z and all other \widehat{r}_i must be matched into their respective \widehat{G}_i° . Furthermore, $\{r, c\}$ and $\{l, x\}$ must be in the matching, giving a unique perfect matching of G .

Second, suppose that all $\widehat{P}(i)$ are false. Then from the unique perfect matchings of each \widehat{G}_k , each \widehat{r}_i must be matched into its respective \widehat{G}_i° . This means $\{z, r\}$, $\{c, l\}$, and $\{x, y\}$ must be in the matching, again giving a unique perfect matching of G . \square

Suppose we have coding graphs G_0, G_1, G_2, \dots such that G_i codes $i \in X$, and that we wish to code $\exists n [\varphi_{e,n}^X(e) \downarrow]$. Unfortunately, our procedure does not have access to the set X itself. Because X is only *coded*, it can only be computed from perfect *matchings* of the G_i .

Lemma 6.10. *There is a uniformly computable procedure which, given any sequence of graphs $\langle G_i \rangle$ coding the predicates $i \in X$ and given any $e \in \mathbb{N}$, returns a single graph that codes the predicate $e \in X'$.*

Proof. As usual, we adopt the convention that $\varphi_{e,n}^X(e) \downarrow$ if and only if $\varphi_{e,n}^{X \upharpoonright n}(e) \downarrow$. Furthermore, we will use the characteristic function of X in the place of X and denote an arbitrary initial segment of X by σ , a finite string of 0's and 1's. Recall that there is a uniformly computable function f such that for each $e, n \in \mathbb{N}$ and each $\sigma \in 2^n$, $\varphi_{e,n}^\sigma(e) = \varphi_{f(e,\sigma),n}(e)$. Thus the statements $\varphi_{f(e,\sigma),n}(e) \downarrow$ clearly have uniformly given coding graphs.

Suppose we wished to perform a computation relative to $X \upharpoonright n$. Without knowing the perfect matchings of the G_i , each string $\sigma \in 2^n$ is a possible initial segment of X . To code $\varphi_{e,n}^X(e) \downarrow$ without any knowledge of X , we code the following statement, which accounts for every possible initial segment σ of X .

$$\bigwedge_{\sigma \in 2^n} \left[\left(\bigwedge_{\sigma(j)=1} (j \in X) \wedge \bigwedge_{\sigma(j)=0} \neg(j \in X) \right) \rightarrow \varphi_{f(e,\sigma),n}(e) \downarrow \right]$$

Note that for each $\sigma \not\prec X$, its corresponding antecedent will be false, so that particular conjunction will be true. For the unique $\sigma \prec X$ of length n , its corresponding antecedent will be true, and the consequent will determine the truth value of the whole conjunction. To determine if $\varphi_e^X(e) \downarrow$, simply code $\exists n [\varphi_{e,n}^X(e) \downarrow]$ as usual. \square

Putting everything together, we obtain the following.

Theorem 6.11. *Let $X \subseteq \mathbb{N}$. For every $e \in \mathcal{O}^X$, there exists an X -computable graph G_e^X such that H_e^X is computable in any perfect matching of G_e^X .*

Proof. Lemmas 6.6 and 6.8 provide a uniform procedure for coding computable sets by sequences of graphs. For successor ordinals, Lemma 6.10 shows that given a sequence of graphs coding a set X , we can uniformly find a sequence of graphs coding X' . For limit ordinals, given any computable simultaneous sequences of graphs $\langle G_{i,n} \rangle$ coding the sets X_n , we can uniformly produce a sequence of graphs coding the effective join $\bigoplus X_n = \{ \langle n, i \rangle : i \in X_n \}$. Here we can simply take $\langle G_{\langle i,n \rangle} \rangle$. By effective transfinite recursion, a

consequence of the recursion theorem, there is therefore a uniformly computable sequence such that for every $e \in \mathcal{O}^X$, there is a graph G_e^X , which is computable in X , such that H_e^X is computable in any perfect matching of G_e^X . \square

Implicit in the above construction is the fact that for each $e \in \mathcal{O}^X$, G_e^X satisfies condition (A). Because we have given a transfinite recursive construction that preserves condition (A) at each successor and limit stage, it follows by transfinite induction that all resulting graphs satisfy condition (A).

Because condition (A) is Π_2^1 , the transfinite induction used above is Π_2^1 -transfinite induction. The system, known as $\Pi_2^1\text{-TI}_0$, is slightly weaker than $\Sigma_2^1\text{-DC}_0$, which is equivalent to $\Delta_2^1\text{-CA}_0$ plus $\Sigma_2^1\text{-IND}$ by Theorem VII.6.9 of [11]. Thankfully, the level of induction can be reduced using the fact that these coding graphs do not have any infinite paths.

Lemma 6.12. *The following hold over $\Pi_1^1\text{-TI}_0$.*

1. *For any e , if $e \in \mathcal{O}^X$, then the coding graph G_e^X has no infinite paths.*
2. *For any e , if $e \in \mathcal{O}^X$, then the coding graph G_e^X satisfies condition (A).*

Proof. The proof of (1) is a straightforward transfinite induction argument. Using (1), the transfinite induction used in (2) becomes Π_1^1 . \square

This gives us the following strengthened result.

Corollary 6.13. *Finite Path PM + $\Pi_1^1\text{-TI}_0$ implies ATR_0 over RCA_0 .*

Proof. This is a direct consequence of Theorem 6.11, together with the above lemma. As PM already implies ACA_0 over RCA_0 , everything can be done over ACA_0 . Fix a set X and an $e \in \mathbb{N}$. Assume $e \in \mathcal{O}^X$. Apply Theorem 6.11 to obtain the coding graph G_e^X . By Lemma 6.12, together with $\Pi_1^1\text{-TI}_0$, we have that G_e^X satisfies condition (A). Then Finite Path PM provides a perfect matching M of G_e^X , from which we can compute H_e^X , again by Theorem 6.11. \square

Although the results above do not give a complete reversal from PM to ATR_0 , they will enable us to separate Finite Path PM, and there for PM, from $\Sigma_1^1\text{-AC}_0$. By Theorem VIII.5.12 of [11], $\Pi_1^1\text{-TI}_0$ is equivalent to $\Sigma_1^1\text{-DC}_0$ over ACA_0 . Then by Theorem VII.6.6 of [11], $\Sigma_1^1\text{-DC}_0$ implies $\Sigma_1^1\text{-AC}_0$. If $\Sigma_1^1\text{-AC}_0$ implied PM, then we would have that $\Sigma_1^1\text{-DC}_0$ implies Finite Path PM, but we show this is not the case.

Corollary 6.14. $\Sigma_1^1\text{-DC}_0$ does not imply Finite Path PM. In particular, PM is strictly stronger than $\Sigma_1^1\text{-AC}_0$.

Proof. Recall that HYP denotes the ω -model consisting of the hyperarithmetical sets. We know that HYP is an ω -model of $\Sigma_1^1\text{-DC}_0$ (by Corollary VIII.4.17 of [11]) but not an ω -model of ATR_0 (by Proposition V.2.6 in [11]). Thus $\Sigma_1^1\text{-DC}_0$ does not imply ATR_0 . But by Corollary 6.13, Finite Path PM plus $\Sigma_1^1\text{-DC}_0$ implies ATR_0 , so it cannot be that $\Sigma_1^1\text{-DC}_0$ implies Finite Path PM. \square

Note that by Corollary 5.4, Collection of PM is equivalent to $\Sigma_1^1\text{-AC}_0$, so neither PM nor Finite Path PM follow from Collection of PM.

Question 6.15. Does Finite Path PM imply ATR_0 over RCA_0 ? This would be true if there is a reversal from Finite Path PM to $\Sigma_1^1\text{-DC}_0$ over $\Sigma_1^1\text{-AC}_0$. Does ATR_0 prove either Finite Path PM or PM?

Our results show that Finite Path PM is quite close in strength to ATR_0 .

Conjecture 6.16. Finite Path PM is equivalent to ATR_0 over RCA_0 .

6.2. Weihrauch reductions

Although it is not the main focus of our paper, the reversal in previous section is perhaps best stated in a special case of the language of Weihrauch reducibility. For a detailed introduction and background, see [16].

In the language of Weihrauch reducibility, mathematical problems are represented by multivalued partial functions, which are written as $f : \subseteq A \rightrightarrows B$, where A, B contain mathematical objects, represented as subsets of \mathbb{N} via the standard codings. For each instance $X \in \text{dom}(f)$, $f(X)$ is the set of all solutions to this instance of the problem. A *realizer* for the problem $f : \subseteq A \rightrightarrows B$ is a function $F : \subseteq A \rightarrow B$ that assigns exactly one solution to each instance of the problem (for each $X \in A$, $F(X)$ is an $F(X) \in f(X)$).

In this language, PM corresponds to the partial multivalued function $f : \subseteq \text{Graphs} \rightrightarrows \text{Matchings}$ whose domain is the set of graphs satisfying condition (A) and so that for each such graph X , $f(X)$ is the set of all perfect matchings of X . A realizer F of this f is any function that assigns a *specific* perfect matching to each $X \in \text{dom}(f)$. To compare PM with ATR_0 , it suffices to use a special case of the Weihrauch Reducibility defined in [16, 14].

Definition 6.17. Let f, g be multivalued functions representing problems in countable mathematics. Then f is Weihrauch reducible to g if there are computable functions $K, H : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that for any G , if G is a realizer for g , then the function defined by $F(p) = K\langle p, G \circ H(p) \rangle$ is a realizer for f .

In [14], Kihara, Marcone and Pauly give a definition for the Weihrauch principle ATR , and show that it is strongly Weihrauch equivalent to the principle “Unique Closed Choice on Baire Space,” written $\text{UC}_{\mathbb{N}^{\mathbb{N}}}$. And while there is no single Weihrauch principle which can be considered the Weihrauch analogue of ATR_0 , the principle $\text{UC}_{\mathbb{N}^{\mathbb{N}}}$, and therefore ATR , is Weihrauch equivalent to many principles which are equivalent to ATR_0 in reverse mathematics.

In particular, we will use a result of Goh [17], that ATR is Weihrauch equivalent to the problem whose instances are pairs (\mathcal{L}, A) such that $\mathcal{L} = (L, 0_L, S, p)$ is a labeled well-ordering and $A \subseteq \mathbb{N}$, and such that for each instance (\mathcal{L}, A) , the unique solution is the jump hierarchy $\langle X_a \rangle_{a \in L}$ starting with A .

Corollary 6.18. *ATR is Weihrauch reducible to Finite Path PM.*

Proof. Consider any pair (\mathcal{L}, A) , where $\mathcal{L} = (L, 0_L, S, p)$ is a labeled well-ordering and $A \subseteq \mathbb{N}$. Perform a similar construction used to prove Theorem 6.11, using $(\mathcal{L} \oplus A)$ -transfinite recursion on L to obtain a graph G such that any perfect matching of G codes the jump hierarchy on L which starts with A . We can use transfinite induction to show that G has no infinite paths, and that G satisfies condition (A). By Finite Path PM, G has a perfect matching M , which codes the jump hierarchy on L starting with A , as desired. \square

Acknowledgements

The authors would like to thank the anonymous referee for many helpful suggestions.

Appendix A. Declarations of interest:

N/A

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