Perfect Matchings in Graphs and Reverse Mathematics

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Springfield College

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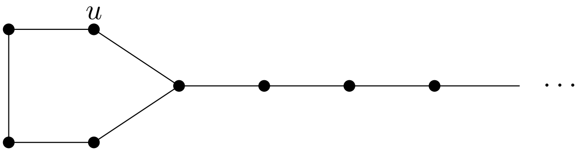
Joint with Stephen Flood, Matthew Jura, and Oscar Levin.

Definition

Let $G = (V, E)$ be a countable graph. A *matching* of G is a set $M \subseteq E$ of edges such that no two distinct edges of M are incident with the same vertex. The *support* of M, denoted by $V(M)$, is the set of matched vertices (i.e., vertices incident with an edge of M).



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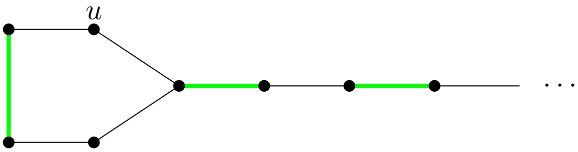

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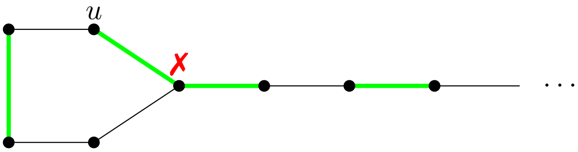

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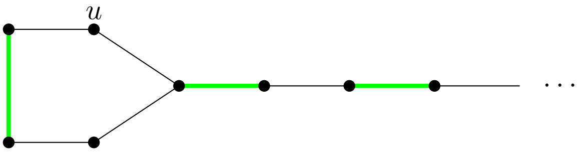

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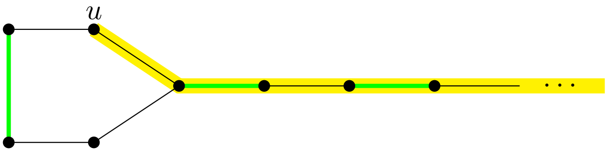

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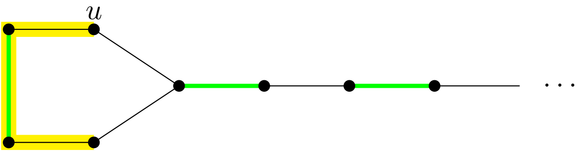

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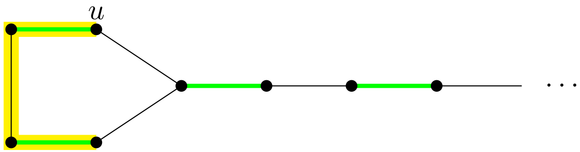

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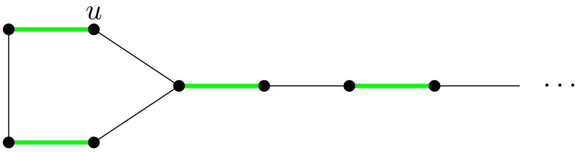

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Condition (A)

A graph $G = (V, E)$ is said to satisfy *condition (A)* if for every matching M and every vertex $s \notin V(M)$ there is an M-augmenting path starting at s.

Theorem

Given $X \subseteq \mathbb{N}$, there is an X-computable graph G satisfying condition (A) such that for every computable ordinal α and every perfect matching M of G, $M \geq_T X^{(\alpha)}$. In fact, G has a unique perfect matching.

Conjecture

The following theorem of Steffens (1977) implies ATR_0, *over* RCA_0*:*

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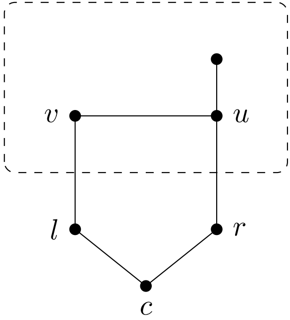

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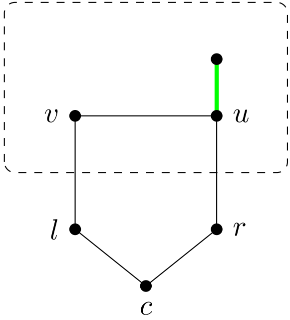

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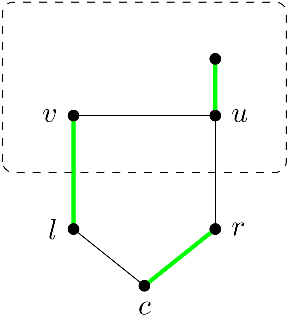

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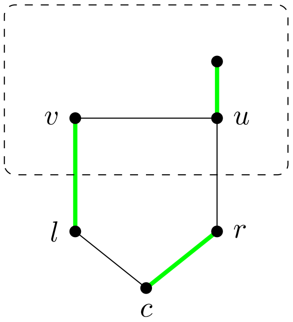

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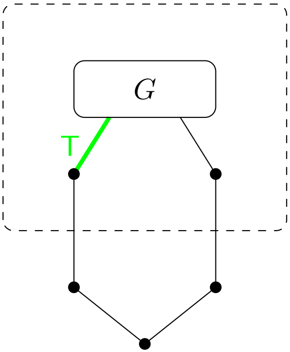

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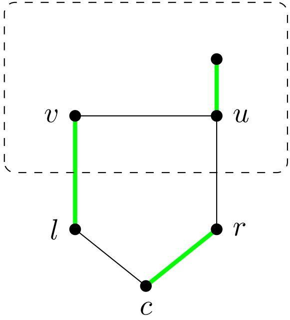

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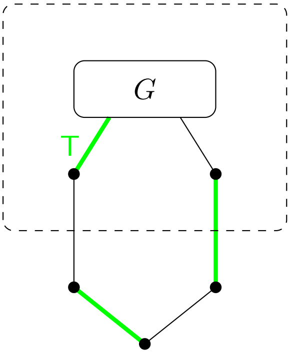

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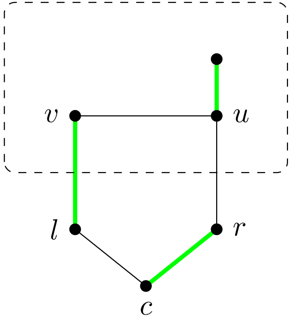

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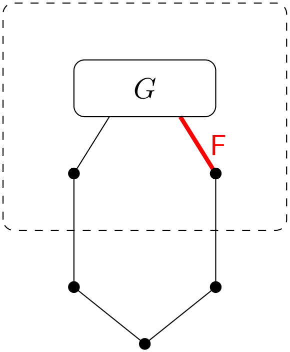

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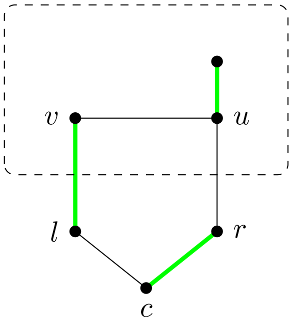

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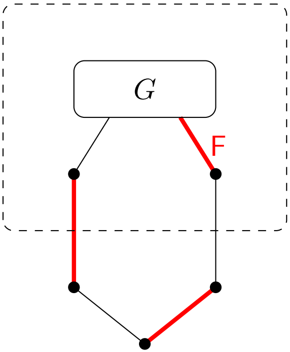

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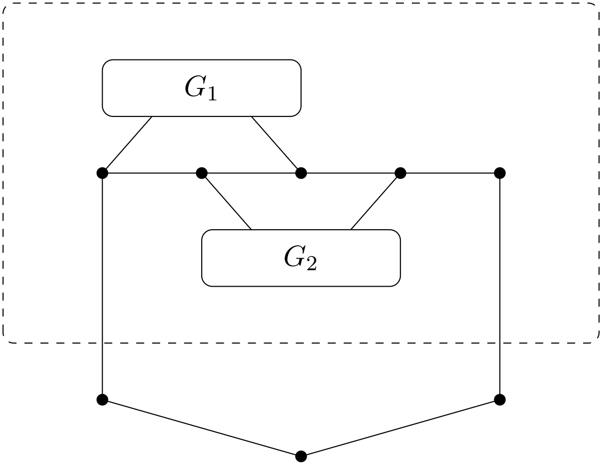

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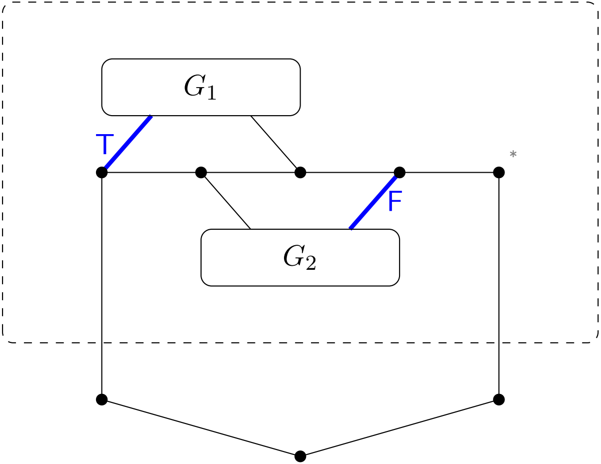

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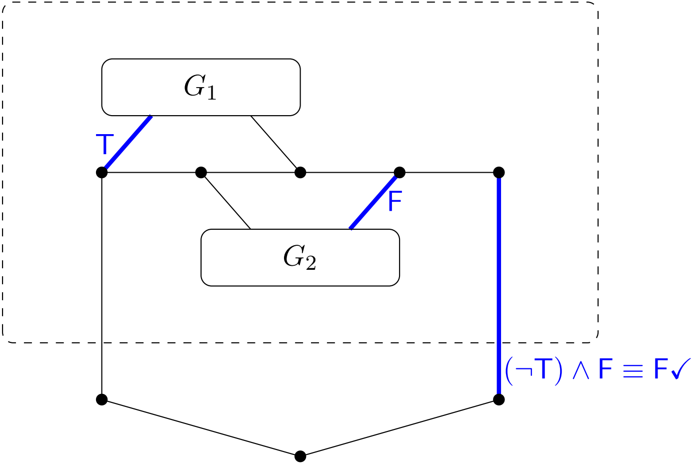

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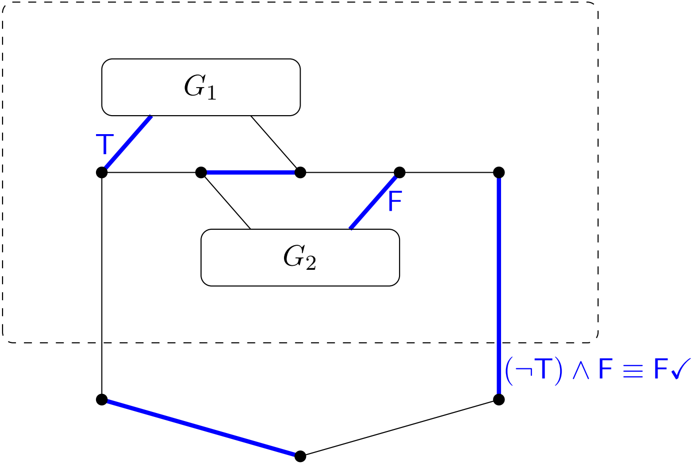

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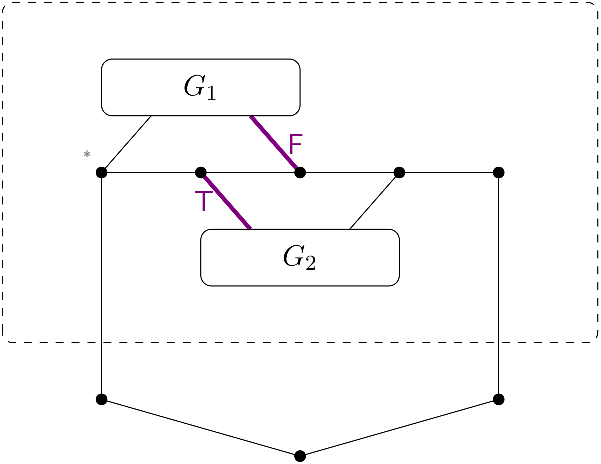

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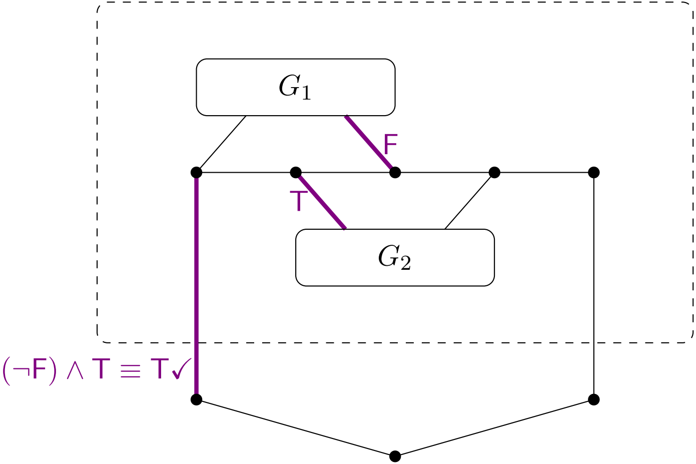

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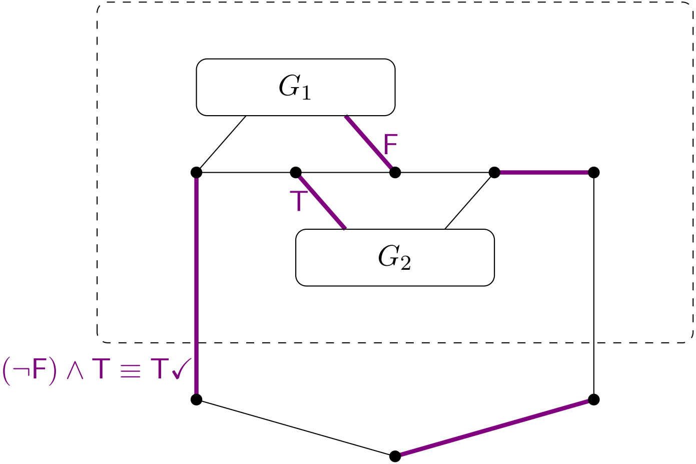
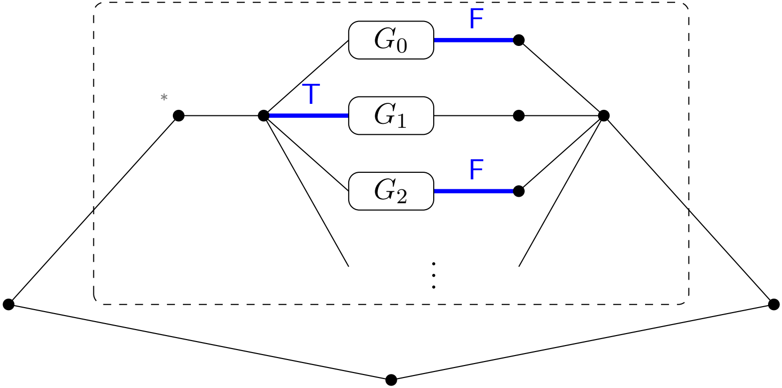


The coding graph for $\exists i\, P(i)$ (assuming at most one $P(i)$ is true), where G_i ($i \in \mathbb{N}$) is the coding graph for $P(i)$



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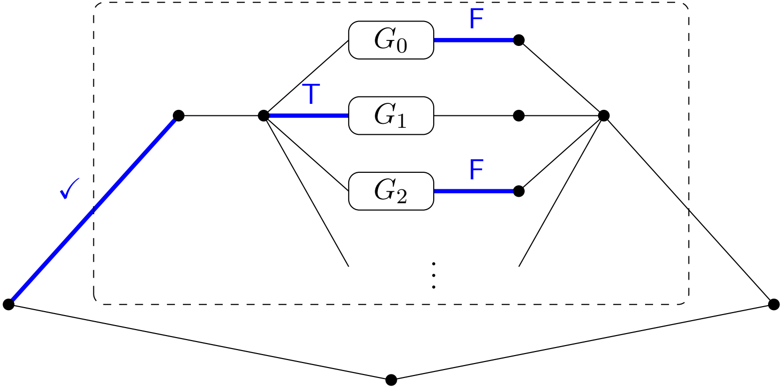

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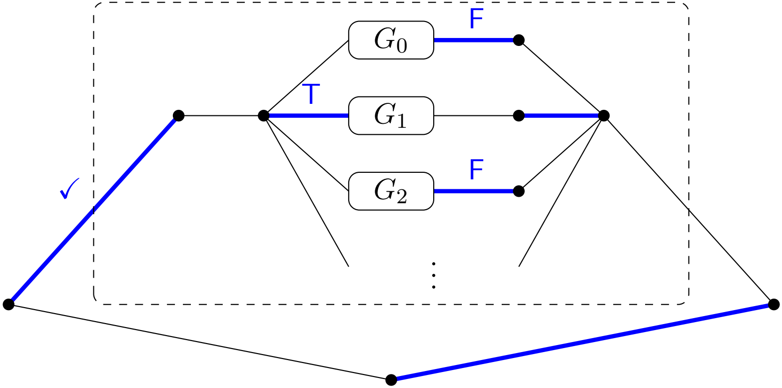

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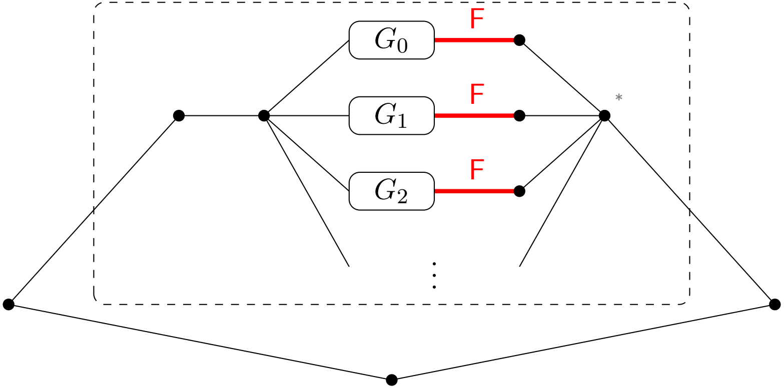

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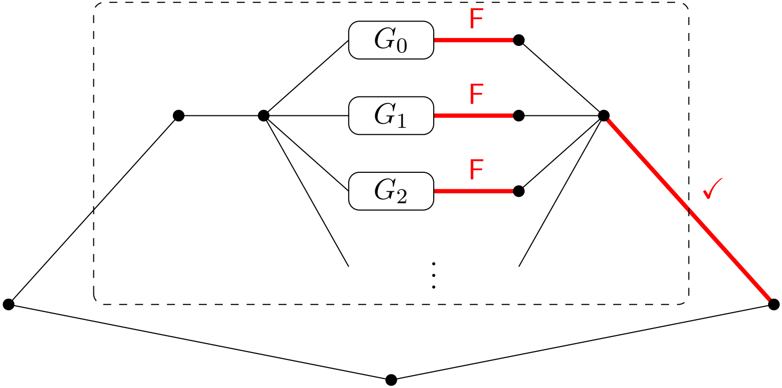

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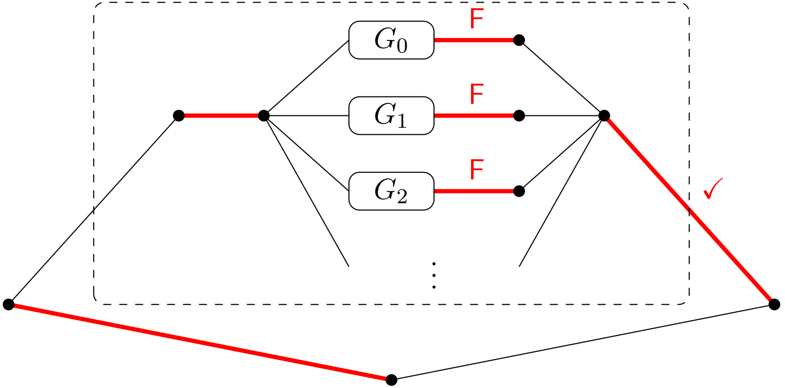

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Further Questions

Theorem (Steffens)

A countable graph G has a perfect matching if and only if G satisfies condition (A), i.e., for every matching M and every vertex $s \notin V(M)$ there is an M-augmenting path starting at s.

1. Does the theorem of Steffens imply a stronger axiom system than ATR_0? More to the point, what is the theorem's exact proof-theoretic strength?
2. What is the strength of (the hard part of) Steffens' proof, involving maximal matchings and independent subgraphs?

Thank you.

R. Aharoni, M. Magidor, and R. A. Shore.
On the strength of König's duality theorem for infinite bipartite graphs.
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K. Steffens.
Matchings in countable graphs.
Can. J. Math., XXIX(1):165–168, 1977.