Cardinality Puzzles

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Definition

We define a **cardinality puzzle** to be any description of a set that explicitly mentions the cardinality of itself.

Puzzle

Let $A = \{2, |A|\}$. What is the cardinality of A?

Solution

Notice that A appears to have two elements. That is, $|A| = 2$. But this observation has the following consequence:

$$|A| = 2 \implies A = \{2, 2\} = \{2\} \implies |A| = 1$$

Let's see what this now leads to:

$$|A| = 1 \implies A = \{2, 1\} \implies |A| = 2$$

We are in a vicious cycle, so this puzzle seems to have no solution.

This is similar to the **Liar Paradox**: "This sentence is false." If it is indeed false, then it's true. And if it's true, then it's false again!

Puzzle

What is the cardinality of the set $A = \{2, 3, |A|\}$?

Solution

It appears to have three elements: $|A| = 3$. But this leads to:

$$|A| = 3 \implies A = \{2, 3, 3\} = \{2, 3\} \implies |A| = 2$$

This in turn leads to:

$$|A| = 2 \implies A = \{2, 3, 2\} = \{2, 3\} \implies |A| = 2$$

That works! So this is not a paradox. It has a solution: the set is $A = \{2, 3\}$ and its cardinality is two.

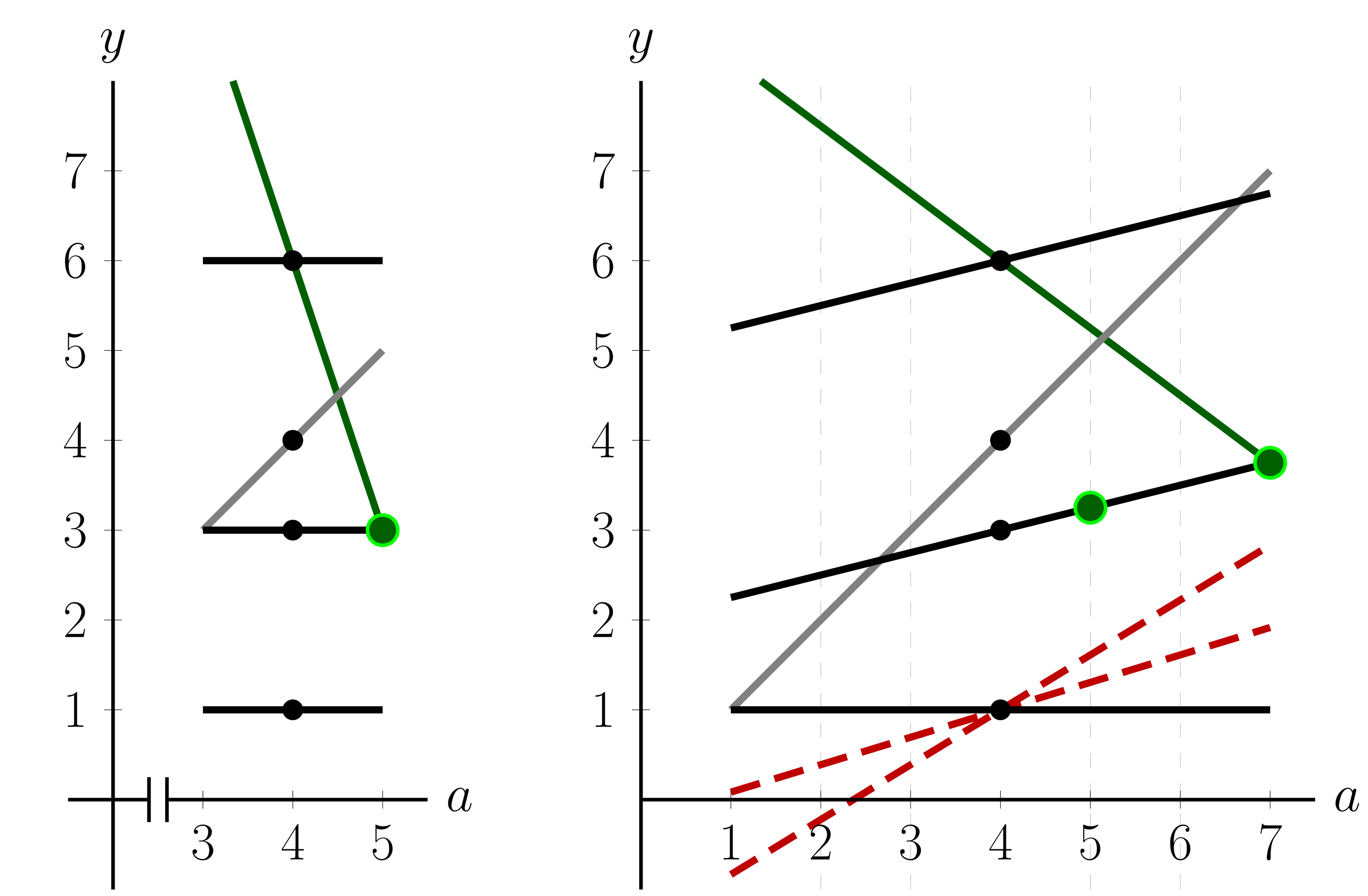
Notation: We'll use a instead of $|A|$. So the above puzzle becomes $A = \{2, 3, a\}$.

Unique Solutions

The two cardinality puzzles $A = \{a, a+1, a+2\}$ and $A = \{4, 5, a, a+1, 2a-1\}$ have the unique solution $\{3, 4, 5\}$. But they have different **complexities**, i.e., number of possible cardinalities.

How To Build Cardinality Puzzles

Given a **selfish set**, say $S = \{1, 3, 4, 6\}$, let's find cardinality puzzles that have S as their unique solution.



$A = \{1, 3, a, 6, 18-3a\}$ $\quad$ $A = \{1, 2+\frac{1}{4}a, a, 5+\frac{1}{4}a, 17-\frac{11}{4}a\}$

Adding more lines increases complexity.

No Solutions (Paradoxes)

Given any natural number $k \geq 1$, the cardinality puzzle $A = \{1, 2, \ldots, k, k+2, a\}$ is a paradox.

Example ($k = 1$)

$A = \{1, 3, a\}$ is a paradox.

Multiple Solutions

Given any natural number $n \geq 2$, the following cardinality puzzle has n solutions: $A = \{n, n+1, n+2, \ldots, 2n-2, a, a+1, \ldots, a+n-1\}$.

Example ($n = 6$)

$A = \{6, 7, 8, 9, 10, a, a+1, \ldots, a+5\}$ has six solutions.

Connection to Knights & Knaves

Knights-and-Knaves Problems: Suppose we encounter some trolls, and each troll is either a knight or a knave. Knights always tell the truth, and knaves always lie. Can we determine which one is which?

- **Troll 1**: We are both knights.
- **Troll 2**: Troll 1 is a knave.

This problem corresponds to the following two-set cardinality puzzle. (There is a method by which we solve the problem that is analogous too!)

Puzzle (Symbiotic Sets)

What are the cardinalities of the sets $A = \{1, a, b\}$ and $B = \{3, a\}$, where $a = |A|$ and $b = |B|$?

Solution

Assume $a = 3$. Then $b = 2$ (based on the elements of A, which show that $b \neq 1$). But at the same time, $B = \{3, 3\} = \{3\}$. So $b = 1$. This is impossible! So $a \neq 3$. Thus, $b = 2$ (based on the elements of B). In the end, $A = \{1, 2, 2\} = \{1, 2\}$ and $B = \{3, 2\}$.

A was a "knave" (it lied about its cardinality), but B was a "knight." This is just like Troll 1 and Troll 2.

Puzzle (Three Sets & Three Trolls)

T1: Only one of us is a knave. $\quad A = \{1, 3, 5, 6, 7, b, c-7\}$

T2: Only one of us is a knight. $\quad B = \{7, 11, a, c\}$

T3: We are all knaves. $\quad C = \{4, 7, 11, 12, \ldots, 16, a, b, c\}$

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